

# Recursive graphical construction of Feynman diagrams in $\phi^4$ theory: Asymmetric case and effective energy

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The free energy of a multicomponent scalar field theory is considered as a functional  $W[G, J]$  of the free correlation function  $G$  and an external current  $J$ . It obeys nonlinear functional differential equations which are turned into recursion relations for the connected Green's functions in a loop expansion. These relations amount to a simple proof that  $W[G, J]$  generates only connected graphs and can be used to find all such graphs with their combinatoric weights. A Legendre transformation with respect to the external current converts the functional differential equations for the free energy into those for the effective energy  $\Gamma[G, \Phi]$ , which is considered as a functional of the free correlation function  $G$  and the field expectation  $\Phi$ . These equations are turned into recursion relations for the one-particle irreducible Green's functions. These relations amount to a simple proof that  $\Gamma[G, J]$  generates only one-particle irreducible graphs and can be used to find all such graphs with their combinatoric weights. The techniques used also allow for a systematic investigation into resummations of classes of graphs. Examples are given for resumming one-loop and multiloop tadpoles, both through all orders of perturbation theory. Since the functional differential equations derived are nonperturbative, they constitute also a convenient starting point for other expansions than those in numbers of loops or powers of coupling constants. We work with general interactions through four powers in the field.

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## I. INTRODUCTION

The free energy of a statistical or quantum field theory may be viewed as a functional of the free correlation functions. It obeys functional differential equations which may be converted into recursion relations for the connected vacuum graphs of the theory. Subsequently, functional derivatives of  $W$  with respect to the free propagators or their inverses can be taken to generate the Feynman diagrams of all connected Green's functions. This program was developed a long time ago by Kleinert [1,2], but used only recently for a systematic generation of all Feynman diagrams of a multicomponent  $\phi^4$  and  $\phi^2A$  theory [3], and of QED [4]. For  $\phi^4$  theory, only the symmetric case was treated.

However, both in statistical physics and particle theory, this symmetry is often broken. For this reason we generalize the symmetric treatment of Ref. [3], and allow for interactions of all powers of the field through four. We introduce an external source  $J$  to be able to generate also Green's functions with odd numbers of external legs as derivatives of  $W$ . In contrast to Ref. [3], this also enables us to generate connected Feynman diagrams for the  $n$ -point functions through  $L$  loops without having to generate any diagrams with more than  $L$  loops first. As a byproduct, we get an alternative proof to the one found in Refs. [1,2] that  $W$  generates only connected Green's functions.

We then Legendre transform the functional differential equations for  $W[G, J]$  into ones for the effective energy (or effective action in quantum theory)  $\Gamma[G, \Phi]$  and derive from these recursion relations for the one-particle irreducible (1PI) Feynman diagrams representing the proper vertices of the theory. No graphs beyond  $L$  loops have to be considered to generate proper  $n$ -point vertices through  $L$  loops. As a by-product, we get an alternative proof that  $\Gamma[G, \Phi]$  gener-

ates only 1 PI Green's functions, similar to the one found in Refs. [1,2].

By using  $G$  as a functional argument, and, to the extent possible, derivatives with respect to  $G$  instead of  $J$  or  $\Phi$ , we keep the identities for  $W$  and  $\Gamma$  and the recursion relations for the connected and 1 PI Green's functions simple. In contrast to Ref. [3], we do not use the technique of "cutting" free correlation functions, but always "amputate" them. As in Ref. [3], the graphical operations necessary to solve the recursion relations can be implemented on a computer for an efficient generation of higher order graphs.

Formally, we consider all our calculations for a statistical theory in  $d$  Euclidean dimensions, but with trivial changes of factors  $i$ , all results are valid as well for a quantum field theory in Minkowski space and for quantum mechanics. In this work, where we often deal with more than one interaction term, our ordering principle is always the number of loops and not powers of coupling constants.

The structure of the paper is as follows. In Sec. II we repeat the steps that led to a functional identity for  $W[G]$  and a recursion relation for its perturbative coefficients in Refs. [1–3]. This gives us the opportunity to specify our slightly different conventions. Going beyond the considerations in Ref. [3], we treat part of the quadratic term as a perturbation. This can be used to cancel one-loop tadpole corrections which drastically reduces the number of vacuum graphs for the free energy, as utilized before in Refs. [5,6].

In Sec. III we treat the asymmetric case for the free energy  $W[G, J]$ . We derive identities for  $W[G, J]$  and recursion relations for the Feynman diagrams representing the connected Green's functions.

In Sec. IV we translate the identities for  $W[G, J]$  into identities for the effective energy  $\Gamma[G, \Phi]$  and subsequently derive recursion relations for the one-particle irreducible

(1 PI) Feynman diagrams representing the proper vertices of the theory. We finally present how part of the quadratic term can be treated as a perturbation to cancel all tadpole corrections to propagators in graphs needed for the proper vertices with one or more external legs, thereby drastically reducing the number of diagrams. Section V contains a summary of our results and an outlook.

## II. SYMMETRIC CASE

### A. Definitions

Consider a scalar field  $\phi$  with  $N$  components in  $d$  Euclidean dimensions whose thermal fluctuations are controlled by the energy functional

$$E[\phi, G, \Delta, L] = \frac{1}{2} \int_{12} (G_{12}^{-1} + \Delta_{12}) \phi_1 \phi_2 + \frac{1}{24} \int_{1234} L_{1234} \phi_1 \phi_2 \phi_3 \phi_4, \quad (1)$$

where  $L_{1234}$  is a self-coupling and where we keep the option open to treat a part  $\Delta_{12}$  of the quadratic term in  $E$  as a perturbation. The numerical indices of  $\int$ ,  $G^{-1}$ ,  $\Delta$ , and  $L$  are meant as a short-hand and represent spatial as well as tensorial arguments

$$1 \equiv \{x_1, \alpha_1\}, \quad \int_1 \equiv \sum_{\alpha_1} \int d^d x_1, \quad \phi_1 \equiv \phi_{\alpha_1}(x_1), \quad (2)$$

$$G_{12}^{-1} \equiv G_{\alpha_1 \alpha_2}^{-1}(x_1, x_2), \quad \Delta_{12} \equiv \Delta_{\alpha_1 \alpha_2}(x_1, x_2),$$

$$L_{1234} \equiv L_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}(x_1, x_2, x_3, x_4). \quad (3)$$

For example, in standard  $\phi^4$  theory we would have

$$G_{\alpha_1 \alpha_2}^{-1}(x_1, x_2) = \delta_{\alpha_1 \alpha_2} \delta(x_1 - x_2) (\partial_1 \cdot \partial_2 + m^2), \quad (4)$$

$$\Delta_{\alpha_1 \alpha_2}(x_1, x_2) = \delta m^2 \delta_{\alpha_1 \alpha_2} \delta(x_1 - x_2), \quad (5)$$

$$L_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}(x_1, x_2, x_3, x_4) = \frac{1}{3} \lambda (\delta_{\alpha_1 \alpha_2} \delta_{\alpha_3 \alpha_4} + \delta_{\alpha_1 \alpha_3} \delta_{\alpha_2 \alpha_4} + \delta_{\alpha_1 \alpha_4} \delta_{\alpha_2 \alpha_3}) \delta(x_1 - x_2) \times \delta(x_1 - x_3) \delta(x_1 - x_4), \quad (6)$$

where  $\delta m^2$  could represent a modification of  $m^2$  that we want to treat perturbatively. Using natural units, where the Boltzmann constant  $k_B$  times the temperature  $T$  equals unity, the partition function  $Z$  and the negative free energy  $W$  are given by a functional integral over the Boltzmann weight  $\exp(-E[\phi])$ ,

$$Z[G, \Delta, L] = \exp(W[G, \Delta, L]) = \int D\phi \exp(-E[\phi, G, \Delta, L]). \quad (7)$$

### B. Identity for $W_I$

Regarding  $W$  as a functional of the kernel  $G^{-1}$ , we can derive a functional differential equation for  $W$ . Our starting point is the identity

$$\int D\phi \frac{\delta}{\delta \phi_1} \{ \phi_2 \exp(-E[\phi, G, \Delta, L]) \} = 0, \quad (8)$$

which follows from functional partial integration and the vanishing of the exponential at infinite fields.

Carrying out the  $\phi_1$  derivative, replacing appearances of  $\phi_i$  with appropriate derivatives with respect to  $G^{-1}$  and finally using the results of Appendix A to translate all such derivatives into derivatives with respect to  $G$  yields an identity for  $W$ ,

$$\begin{aligned} \delta_{12} - 2 \int_3 G_{23} \frac{\delta W}{\delta G_{13}} - 2 \int_{345} \Delta_{13} G_{24} G_{35} \frac{\delta W}{\delta G_{45}} \\ - \frac{2}{3} \int_{345} L_{1345} \left[ \int_{67} (G_{23} G_{46} G_{57} + G_{26} G_{37} G_{45}) \frac{\delta W}{\delta G_{67}} \right] \\ - \frac{2}{3} \int_{345} L_{1345} \left[ \int_{6789} G_{26} G_{37} G_{48} G_{59} \frac{\delta^2 W}{\delta G_{67} \delta G_{89}} \right] \\ - \frac{2}{3} \int_{345} L_{1345} \left[ \int_{6789} G_{26} G_{37} \frac{\delta W}{\delta G_{67}} G_{48} G_{59} \frac{\delta W}{\delta G_{89}} \right] = 0. \end{aligned} \quad (9)$$

Split  $W$  into a free and an interacting part,

$$W = W_0 + W_I \equiv W|_{\Delta, L=0} + W_I. \quad (10)$$

For  $W_0$ , Eq. (9) reduces to

$$\delta_{12} - 2 \int_3 G_{23} \frac{\delta W_0}{\delta G_{13}} = 0, \quad (11)$$

so that, using also the results of Appendix A, we get the useful relations

$$\frac{\delta W_0}{\delta G_{12}} = \frac{1}{2} G_{12}^{-1} \quad (12)$$

and

$$\frac{\delta^2 W_0}{\delta G_{12} \delta G_{34}} = -\frac{1}{4} (G_{13}^{-1} G_{24}^{-1} + G_{14}^{-1} G_{23}^{-1}). \quad (13)$$

Up to an additive constant, which we assume to be adjusted to zero by an appropriate normalization of the path integral measure  $D\phi$ ,  $W_0$  itself is given as usual by

$$W_0[G] = -\frac{1}{2} \int_1 (\ln G^{-1})_{11} = \frac{1}{2} \bigcirc, \quad (14)$$

where we have introduced a graphical representation for  $W_0$ .

Subtracting Eq. (11) from Eq. (9), using Eqs. (12) and (13), setting  $x_2 = x_1$ , and integrating over  $x_1$  gives a nonlinear functional differential equation for  $W_I$ ,

$$\begin{aligned}
 & \int_{12} G_{12} \frac{\delta W_I}{\delta G_{12}} + \frac{1}{4} \int_{1234} L_{1234} G_{12} G_{34} + \frac{1}{2} \int_{12} \Delta_{12} G_{12} \\
 & + \int_{1234} \Delta_{12} G_{13} G_{24} \frac{\delta W_I}{\delta G_{34}} + \int_{123456} L_{1234} G_{12} G_{35} G_{46} \frac{\delta W_I}{\delta G_{56}} \\
 & + \frac{1}{3} \int_{12345678} L_{1234} G_{15} G_{26} G_{37} G_{48} \frac{\delta^2 W_I}{\delta G_{56} \delta G_{78}} \\
 & + \frac{1}{3} \int_{12345678} L_{1234} G_{15} G_{26} G_{37} G_{48} \frac{\delta W_I}{\delta G_{56}} \frac{\delta W_I}{\delta G_{78}} = 0. \quad (15)
 \end{aligned}$$

$$\frac{\delta W_I}{\delta G_{12}} = \begin{array}{c} 1 \\ \circlearrowleft \\ 2 \end{array} W_I, \quad \frac{\delta^2 W_I}{\delta G_{12} \delta G_{34}} = \begin{array}{c} 1 \\ \circlearrowleft \\ 2 \\ \circlearrowleft \\ 3 \\ \circlearrowleft \\ 4 \end{array} W_I \quad (16)$$

$$-\Delta_{12} = 1 \text{---} \square \text{---} 2, \quad -L_{1234} = \begin{array}{c} 1 \\ \times \\ 2 \\ \times \\ 3 \\ \times \\ 4 \end{array}. \quad (17)$$

For  $\Delta=0$ , this reduces to Eq. (2.58) in Ref. [3].

Lines that are connected at both ends are propagators  $G$ . All space arguments that are not indicated by numbers are integrated over. Now Eq. (15) reads

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$$\begin{array}{c} \circlearrowleft \\ W_I \end{array} = \frac{1}{4} \begin{array}{c} \circlearrowleft \\ \circlearrowleft \end{array} + \frac{1}{2} \begin{array}{c} \square \\ \square \end{array} + \begin{array}{c} \square \\ \square \\ \circlearrowleft \\ W_I \end{array} + \begin{array}{c} \circlearrowleft \\ \circlearrowleft \\ W_I \end{array} + \frac{1}{3} \begin{array}{c} \circlearrowleft \\ \circlearrowleft \\ \circlearrowleft \\ W_I \end{array} + \frac{1}{3} \begin{array}{c} \circlearrowleft \\ \circlearrowleft \\ \circlearrowleft \\ W_I \end{array} \begin{array}{c} \circlearrowleft \\ \circlearrowleft \\ W_I \end{array}. \quad (18)$$

Note that a derivative with respect to  $G$  graphically means removing (“amputating”) a line from a Feynman graph (for details see Ref. [3] and Appendix A). This will be important when we represent  $W_I$  as a sum of Feynman graphs in the next section. For example, the operation on  $W_I$  on the left-hand side of Eq. (18) multiplies each graph in  $W_I$  by the number of its lines.

$$-\Delta \equiv \text{---} \square \text{---} = - \sum_{L=1}^{\infty} \Delta^{(L)} \equiv \sum_{L=1}^{\infty} \text{---} \square \text{---}, \quad (20)$$

where  $\Delta^{(L)}$  counts formally for  $L$  intrinsic loops.

Equation (18) then splits into

### C. Recursion relation

Now split  $W_I$  and  $\Delta$  into different loop orders,

$$\begin{array}{c} \circlearrowleft \\ 2 \end{array} = \frac{1}{4} \begin{array}{c} \circlearrowleft \\ \circlearrowleft \end{array} + \frac{1}{2} \begin{array}{c} \square \\ \square \end{array}, \quad (21)$$

from which follows

$$W_I \equiv \begin{array}{c} \circlearrowleft \\ W_I \end{array} = \sum_{L=2}^{\infty} W^{(L)} \equiv \sum_{L=2}^{\infty} \begin{array}{c} \circlearrowleft \\ L \end{array}, \quad (19)$$

$$\begin{array}{c} \circlearrowleft \\ 2 \end{array} = \frac{1}{8} \begin{array}{c} \circlearrowleft \\ \circlearrowleft \end{array} + \frac{1}{2} \begin{array}{c} \square \\ \square \end{array}, \quad (22)$$

and

and

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$$\begin{array}{c} \circlearrowleft \\ L \end{array} = \frac{1}{2} \begin{array}{c} \square \\ \square \end{array} + \sum_{l=1}^{L-2} \begin{array}{c} \square \\ \square \\ \circlearrowleft \\ L-l \end{array} + \begin{array}{c} \circlearrowleft \\ \circlearrowleft \\ L-1 \end{array} + \frac{1}{3} \begin{array}{c} \circlearrowleft \\ \circlearrowleft \\ \circlearrowleft \\ L-1 \end{array} + \frac{1}{3} \sum_{l=2}^{L-2} \begin{array}{c} \circlearrowleft \\ \circlearrowleft \\ \circlearrowleft \\ l \end{array} \begin{array}{c} \circlearrowleft \\ \circlearrowleft \\ L-l \end{array} \quad (23)$$

for  $L > 2$ .

Let us now derive a recursion relation for the  $W^{(L)}$  themselves instead of their derivatives with respect to  $G$ . First note that since  $W$  depends only on  $G^{-1}$  and  $\Delta$  only through the combination  $G^{-1} + \Delta$  we have

$$\frac{\delta W}{\delta \Delta_{12}} = \frac{\delta W}{\delta G_{12}^{-1}} = - \int_{34} G_{13} G_{24} \frac{\delta W}{\delta G_{34}}, \quad (24)$$

where we have used Eq. (A12).

Because of Eq. (20) we can write then for any  $L$

$$\int_{12} \Delta_{12}^{(L)} \frac{\delta W}{\delta \Delta_{12}^{(L)}} = - \int_{1234} G_{31} \Delta_{12}^{(L)} G_{24} \frac{\delta W}{\delta G_{34}} \equiv \boxed{L} \text{ (W)}. \quad (25)$$

Splitting up into loop orders gives

$$\int_{12} \Delta_{12}^{(l)} \frac{\delta W^{(L)}}{\delta \Delta_{12}^{(l)}} = - \int_{1234} \Delta_{12}^{(l)} G_{13} G_{24} \frac{\delta W^{(L-l)}}{\delta G_{34}} \equiv \boxed{l} \text{ (L-l)} \quad (26)$$

for  $1 \leq l \leq L-2$  and, using Eq. (12),

$$\int_{12} \Delta_{12}^{(L-1)} \frac{\delta W^{(L)}}{\delta \Delta_{12}^{(L-1)}} = - \int_{1234} \Delta_{12}^{(L-1)} G_{13} G_{24} \frac{\delta W_0}{\delta G_{34}} = -\frac{1}{2} \int_{12} \Delta_{12}^{(L-1)} G_{12} \equiv \frac{1}{2} \boxed{L-1} \quad (27)$$

for  $L \geq 1$ . Since an  $L$ -loop diagram without two-point insertions contains  $2(L-1)$  propagators and since an  $l$ -loop two-point insertion causes a reduction in the number of propagators by  $2l-1$ , the following relation for the  $L$ -loop contribution to  $W$  holds for  $L \geq 2$ :

$$\int_{12} \left[ G_{12} \frac{\delta}{\delta G_{12}} + \sum_{l=1}^{L-1} (2l-1) \Delta_{12}^{(l)} \frac{\delta}{\delta \Delta_{12}^{(l)}} \right] W^{(L)} = 2(L-1) W^{(L)}. \quad (28)$$

Making use of Eqs. (26) and (27) this can be rewritten as

$$\left[ \int_{12} G_{12} \frac{\delta W^{(L)}}{\delta G_{12}} - \frac{2L-3}{2} \int_{12} \Delta_{12}^{(L-1)} G_{12} - \sum_{l=1}^{L-2} (2l-1) \int_{123} \Delta_{12}^{(l)} G_{13} G_{24} \frac{\delta W^{(L-l)}}{\delta G_{34}} \right] = 2(L-1) W^{(L)} \quad (29)$$

or

$$\text{Diagram } L + \frac{2L-3}{2} \boxed{L-1} + \sum_{l=1}^{L-2} (2l-1) \boxed{l} \text{ (L-l)} = 2(L-1) \text{Diagram } L. \quad (30)$$

Rewriting the first term using the recursion relation (23) gives

$$\begin{aligned} \text{Diagram } L &= \frac{1}{2} \boxed{L-1} \\ &+ \frac{1}{L-1} \left[ \frac{1}{6} \text{Diagram } (L-1) + \sum_{l=1}^{L-2} l \boxed{l} \text{ (L-l)} + \frac{1}{2} \text{Diagram } (L-1) + \frac{1}{6} \sum_{l=2}^{L-2} \text{Diagram } (l) \text{ (L-l)} \right] \end{aligned} \quad (31)$$

for  $L > 2$ . For  $\Delta = 0$  and appropriately adjusted conventions this reduces to Eq. (2.64) in [3].

We have used Eqs. (22) and (31) to determine all vacuum graphs and their weights (i.e., combinatorial prefactors) through five loops for the case  $\Delta = 0$  and listed them in Table I.

With a one-loop correction

$$\Delta = \Delta^{(1)} = \text{Diagram } \boxed{1} \equiv \text{Diagram } \bullet \quad (32)$$

we get the additional graphs listed in Table II.

### D. One-loop resummation

Let us now try to adjust the one-loop two-point insertion (32) so that it cancels the trivial but ubiquitous one-loop fluctuation  $\text{Diagram } \text{O}$ , present in most diagrams in Table I. For this purpose, define

TABLE I. Vacuum diagrams with their weights through five loops.

Number of loops	Diagrams with their weights
1	$\frac{1}{2}$
2	$\frac{1}{8}$
3	$\frac{1}{48}$ $\frac{1}{16}$
4	$\frac{1}{48}$ $\frac{1}{24}$ $\frac{1}{32}$ $\frac{1}{48}$
5	$\frac{1}{128}$ $\frac{1}{144}$ $\frac{1}{32}$ $\frac{1}{16}$ $\frac{1}{48}$ $\frac{1}{32}$ $\frac{1}{48}$ $\frac{1}{64}$ $\frac{1}{32}$ $\frac{1}{128}$

$$\text{---}\textcircled{1}\text{---} = \text{---}\textcircled{1}\text{---} + \frac{1}{2} \text{---}\textcircled{\text{---}}\text{---} \tag{33}$$

and set

$$\text{---}\textcircled{L}\text{---} = 0 \tag{34}$$

for  $L > 1$ . Then Eq. (31) becomes

$$\textcircled{L} = \frac{1}{L-1} \left[ \frac{1}{6} \textcircled{\textcircled{L-1}} + \textcircled{1}\textcircled{L-1} + \frac{1}{6} \sum_{l=2}^{L-2} \textcircled{l}\textcircled{L-l} \right] \tag{35}$$

for  $L > 2$ .

Now we show (i) that  $W^{(3)}$  contains the two terms on the right hand side of Eq. (33) only in this combination and (ii) that if  $W^{(L)}$  with  $L > 2$  contains the two terms on the right-hand side of Eq. (33) only in this combination, then this is also true for  $W^{(L+1)}$ .

Using Eqs. (22) and (35) gives

$$\textcircled{3} = \frac{1}{48} \textcircled{\textcircled{2}} + \frac{1}{4} \textcircled{1}\textcircled{1}, \tag{36}$$

which proves (i). The only terms on the right hand side of Eq. (35) that could potentially violate (ii) are the terms with  $l = 2$  and/or  $L-l=2$  in the sum. If both  $l=2$  and  $L-l=2$  (i.e., for  $L=4$ ), the only term in the sum is

TABLE II. Additional vacuum diagrams through five loops caused by the one-loop insertion  $\boxed{1}$   $\equiv$   $\bullet$

and their weights.

Number of loops	Additional diagrams with their weights
2	$\frac{1}{2}$
3	$\frac{1}{4}$ $\frac{1}{4}$
4	$\frac{1}{6}$ $\frac{1}{8}$ $\frac{1}{4}$ $\frac{1}{12}$ $\frac{1}{8}$ $\frac{1}{8}$
5	$\frac{1}{8}$ $\frac{1}{4}$ $\frac{1}{4}$ $\frac{1}{12}$ $\frac{1}{8}$ $\frac{1}{16}$ $\frac{1}{8}$ $\frac{1}{8}$ $\frac{1}{16}$ $\frac{1}{4}$ $\frac{1}{8}$ $\frac{1}{8}$ $\frac{1}{12}$ $\frac{1}{24}$ $\frac{1}{8}$ $\frac{1}{16}$ $\frac{1}{16}$ $\frac{1}{16}$

$$\begin{aligned}
 \text{Diagram with two circles and a dot} &= \frac{1}{16} \text{Diagram 1} + \frac{1}{8} \text{Diagram 2} + \frac{1}{8} \text{Diagram 3} + \frac{1}{4} \text{Diagram 4} \\
 &= \frac{1}{4} \text{Diagram 5} .
 \end{aligned}
 \tag{37}$$

If only one of  $l$  and  $L-l$  equals 2 (i.e., for  $L > 4$ ), the potentially dangerous terms in the sum are of the form

$$\text{Diagram with circles 2 and L-2} = \frac{1}{4} \text{Diagram 1} + \frac{1}{2} \text{Diagram 2} = \frac{1}{2} \text{Diagram 3} .
 \tag{38}$$

That is, in both cases only the combination on the right-hand side of Eq. (33) appears. This finishes the proof of (ii).

Once we have used the recursion relation (35) to compute any  $W^{(L)}$ , we can set

$$\text{Diagram with a dot in a box} = 0 .
 \tag{40}$$

$$\text{Diagram with a dot in a box} = -\frac{1}{2} \text{Diagram with a dot}
 \tag{39}$$

This drastically reduces the number of diagrams in any given order, since for  $L > 2$  no one-loop mass corrections is present anymore. However, we are not allowed to use the result as an input for our recursion relation (35), since the two terms on the right hand side of Eq. (33) behave differently in the recursion relation.

Note that now

such that

TABLE III. Remaining diagrams with their weights through six loops with a one-loop adjusted two-point insertion.

Number of loops	Remaining diagrams with their weights
1,2,3,4	$\frac{1}{2}$ $-\frac{1}{8}$ $\frac{1}{48}$ $\frac{1}{48}$
5	$\frac{1}{128}$ $\frac{1}{144}$ $\frac{1}{32}$
6	$\frac{1}{320}$ $\frac{1}{288}$ $\frac{1}{48}$ $\frac{1}{32}$ $\frac{1}{16}$ $\frac{1}{120}$

$$\textcircled{2} = -\frac{1}{8} \text{figure-eight} + \frac{1}{2} \textcircled{1} . \quad (41)$$

universal critical amplitude ratios [7].

As an alternative to explicitly constructing the graphs in Table II by recursion relations, we could replace the propagator in the graphs of Table I according to

With the condition (40) this becomes

$$G^{-1} \rightarrow G^{-1} + \Delta, \quad (43)$$

$$\textcircled{2} = -\frac{1}{8} \text{figure-eight}, \quad (42)$$

i.e.,

$$G \rightarrow (G^{-1} + \Delta)^{-1} = G(1 + \Delta G)^{-1} = G + G\Delta G + G\Delta G\Delta G + \dots, \quad (44)$$

which is the only diagram left with a one-loop mass correction.

with

In Table III we list the diagrams with their weights through six loops that are left after adjusting the two-point insertion according to Eq. (39). Through four and five loops this adjustment has been used in Refs. [5] and [6], respectively, to simplify the renormalization of the vacuum energy in  $\phi^4$  theory, which is used for the computation of some

$$\Delta_{12} = -\frac{1}{2} \int_{34} L_{1234} G_{34}. \quad (45)$$

Diagrammatically, this amounts to replacing

$$\text{---} \rightarrow \text{---} - \frac{1}{2} \text{---} \textcircled{1} \text{---} + \frac{1}{4} \text{---} \textcircled{1} \textcircled{1} \text{---} - \frac{1}{8} \text{---} \textcircled{1} \textcircled{1} \textcircled{1} \text{---} + \dots \quad (46)$$

and adding up the resulting graphs through the appropriate loop order. The result is again the graphs in Table III with the same weights.

### III. GENERAL CASE

#### A. Definitions

Now let us generalize our treatment to the case with general interactions through four powers in the field

$$E[\phi, C, J, G, K, L] = C + \int_1 J_1 \phi_1 + \frac{1}{2} \int_{12} G_{12}^{-1} \phi_1 \phi_2 + \frac{1}{6} \int_{123} K_{123} \phi_1 \phi_2 \phi_3 + \frac{1}{24} \int_{1234} L_{1234} \phi_1 \phi_2 \phi_3 \phi_4, \quad (47)$$

where  $G_{12}^{-1}$ ,  $K_{123}$ , and  $L_{1234}$  are symmetric in their indices. For example, for a  $Z_2$ -symmetric single-component  $\phi^4$  theory with background field  $\phi$ , i.e.,

$$E[\phi] = \int_1 \left[ \frac{1}{2} (\partial_\mu \phi + \partial_\mu \phi)^2 + \frac{1}{2} m^2 (\phi + \phi)^2 + \frac{1}{24} \lambda (\phi + \phi)^4 + c \right]_1, \quad (48)$$

we would have

$$C = \int_1 \left[ \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{24} \lambda \phi^4 + c \right]_1, \quad (49)$$

$$J_1 = -\partial^2 \phi_1 + m^2 \phi_1 + \frac{1}{6} \lambda \phi_1^3, \quad (50)$$

$$G_{12}^{-1} = \delta_{12} (\partial_1 \cdot \partial_2 + m^2 + \frac{1}{2} \lambda \phi_1 \phi_2), \quad (51)$$

$$K_{123} = \delta_{12} \delta_{13} \lambda \phi_1, \quad (52)$$

$$L_{1234} = \delta_{12} \delta_{13} \delta_{14} \lambda. \quad (53)$$

The partition function  $Z$  and the negative free energy  $W$  are given by

$$\begin{aligned} Z[C, J, G, K, L] &= \exp(W[C, J, G, K, L]) \\ &= \int D\phi \exp(-E[\phi, C, J, G, K, L]). \end{aligned} \quad (54)$$

The energy is now regarded as a function of  $C$  and a functional of  $\phi$ ,  $J$ ,  $G$ ,  $K$ , and  $L$ . We will mainly be interested in its dependence on  $\phi$ ,  $J$ , and  $G$ .

#### B. Identities for $W_I$

We continue with deriving identities similar to Eq. (9). We now have the possibility to represent each occurrence of the field  $\phi$  by a derivative with respect to  $J$ . We keep the number of these derivatives at a minimum and use as much as possible derivatives with respect to  $G$  to keep the identities and the recursion relations derived from them as simple as possible.

The identities we need are

$$\begin{aligned} 0 &= \exp(-W[C, J, G, K, L]) \int D\phi \frac{\delta}{\delta \phi_1} \\ &\quad \times \exp(-E[\phi, C, J, G, K, L]) \\ &= -J_1 + \int_2 G_{12}^{-1} \frac{\delta W}{\delta J_2} + \int_{23} K_{123} \frac{\delta W}{\delta G_{23}^{-1}} \\ &\quad - \frac{1}{3} \int_{234} L_{1234} \left( \frac{\delta^2 W}{\delta J_2 \delta G_{34}^{-1}} + \frac{\delta W}{\delta J_2} \frac{\delta W}{\delta G_{34}^{-1}} \right) \end{aligned} \quad (55)$$

and

$$\begin{aligned} 0 &= \exp(-W[C, J, G, K, L]) \int D\phi \frac{\delta}{\delta \phi_1} \\ &\quad \times \{ \phi_2 \exp(-E[\phi, C, J, G, K, L]) \} \\ &= \delta_{12} + J_1 \frac{\delta W}{\delta J_2} + 2 \int_3 G_{13}^{-1} \frac{\delta W}{\delta G_{23}^{-1}} \\ &\quad - \int_{34} K_{134} \left( \frac{\delta^2 W}{\delta J_2 \delta G_{34}^{-1}} + \frac{\delta W}{\delta J_2} \frac{\delta W}{\delta G_{34}^{-1}} \right) \\ &\quad - \frac{2}{3} \int_{345} L_{1345} \left( \frac{\delta^2 W}{\delta G_{23}^{-1} \delta G_{45}^{-1}} + \frac{\delta W}{\delta G_{23}^{-1}} \frac{\delta W}{\delta G_{45}^{-1}} \right), \end{aligned} \quad (56)$$

where we have followed similar steps as for the derivation of Eq. (9) except that we have not yet replaced derivatives with respect to  $G^{-1}$  by those with respect to  $G$ .

Split  $W$  again into a free and an interacting part,

$$W = W_0 + W_I \equiv W|_{K,L=0} + W_I. \quad (57)$$

For  $W_0$ , Eqs. (55) and (56) reduce to

$$-J_1 + \int_2 G_{12}^{-1} \frac{\delta W_0}{\delta J_2} = 0 \quad (58)$$

and

$$\delta_{12} + J_1 \frac{\delta W_0}{\delta J_2} + 2 \int_3 G_{13}^{-1} \frac{\delta W_0}{\delta G_{23}^{-1}} = 0, \quad (59)$$

respectively. Combining them and using the results from Appendix A, we get the useful relations

$$\frac{\delta W_0}{\delta J_1} = \int_2 G_{12} J_2, \quad (60)$$

$$\frac{\delta W_0}{\delta G_{12}^{-1}} = -\frac{1}{2} \left( G_{12} + \int_{34} G_{13} J_3 G_{24} J_4 \right), \quad (61)$$

$$\frac{\delta^2 W_0}{\delta J_1 \delta G_{23}^{-1}} = -\frac{1}{2} \int_4 (G_{12} G_{34} + G_{13} G_{24}) J_4, \quad (62)$$

and



$$\frac{\delta^2 W_0}{\delta G_{12}^{-1} \delta G_{34}^{-1}} = \frac{1}{4} \left[ G_{13} G_{24} + G_{14} G_{23} + \int_{56} (G_{13} G_{25} G_{46} + G_{14} G_{25} G_{36} + G_{23} G_{15} G_{46} + G_{24} G_{15} G_{36}) J_5 J_6 \right]. \quad (63)$$

With the same normalization of the path integral measure  $D\phi$  as before we have

$$\begin{aligned} W_0[C, J, G] &= \ln \int D\phi \exp \left( -C - \int_1 J_1 \phi_1 - \frac{1}{2} \int_{12} G_{12}^{-1} \phi_1 \phi_2 \right) \\ &= \ln \int D\phi \exp \left[ -C - \int_1 J_1 \left( \phi_1 - \int_2 G_{12} J_2 \right) - \frac{1}{2} \int_{12} G_{12}^{-1} \left( \phi_1 - \int_3 G_{13} J_3 \right) \left( \phi_2 - \int_4 G_{24} J_4 \right) \right] \\ &= -C + \frac{1}{2} \int_{12} G_{12} J_1 J_2 + \ln \int D\phi \exp \left( -\frac{1}{2} \int_{12} G_{12}^{-1} \phi_1 \phi_2 \right) = -C + \frac{1}{2} \int_{12} G_{12} J_1 J_2 - \frac{1}{2} \int_1 (\ln G^{-1})_{11}. \quad (64) \end{aligned}$$

Subtracting Eq. (58) from Eq. (55), multiplying with  $\int_2 G_{12} J_2$ , integrating over  $x_1$  and using Eqs. (60)–(62) and Eq. (A12), we get

$$\begin{aligned} & \int_1 J_1 \frac{\delta W_I}{\delta J_1} - \frac{1}{2} \int_{1234} K_{123} G_{12} G_{34} J_4 - \frac{1}{2} \int_{123456} K_{123} G_{14} J_4 G_{25} J_5 G_{36} J_6 + \frac{1}{2} \int_{123456} L_{1234} G_{12} G_{35} J_5 G_{46} J_6 \\ & + \frac{1}{6} \int_{12345678} L_{1234} G_{15} J_5 G_{26} J_6 G_{37} J_7 G_{48} J_8 - \int_{123456} K_{123} G_{14} J_4 G_{25} G_{36} \frac{\delta W_I}{\delta G_{56}} + \frac{1}{6} \int_{12345} L_{1234} G_{12} G_{35} J_5 \frac{\delta W_I}{\delta J_4} \\ & + \frac{1}{6} \int_{1234567} L_{1234} G_{15} J_5 G_{26} J_6 G_{37} J_7 \frac{\delta W_I}{\delta J_4} + \frac{1}{3} \int_{12345678} L_{1234} G_{15} J_5 G_{26} J_6 G_{37} G_{48} \frac{\delta W_I}{\delta G_{78}} \\ & + \frac{1}{3} \int_{1234567} L_{1234} G_{15} J_5 G_{36} G_{47} \frac{\delta^2 W_I}{\delta J_2 \delta G_{67}} + \frac{1}{3} \int_{1234567} L_{1234} G_{15} J_5 G_{36} G_{47} \frac{\delta W_I}{\delta J_2} \frac{\delta W_I}{\delta G_{67}} = 0. \quad (65) \end{aligned}$$

Subtracting Eq. (59) from Eq. (56), setting  $x_2 = x_1$ , integrating over  $x_1$  and using Eqs. (60)–(63) and (A12) we get

$$\begin{aligned} & \int_1 J_1 \frac{\delta W_I}{\delta J_1} - 2 \int_{12} G_{12} \frac{\delta W_I}{\delta G_{12}} + \frac{3}{2} \int_{1234} K_{123} G_{12} G_{34} J_4 + \frac{1}{2} \int_{123456} K_{123} G_{14} J_4 G_{25} J_5 G_{36} J_6 - \frac{1}{2} \int_{1234} L_{1234} G_{12} G_{34} \\ & - \int_{123456} L_{1234} G_{12} G_{35} J_5 G_{46} J_6 - \frac{1}{6} \int_{12345678} L_{1234} G_{15} J_5 G_{26} J_6 G_{37} J_7 G_{48} J_8 + \frac{1}{2} \int_{123} K_{123} G_{12} \frac{\delta W_I}{\delta J_3} \\ & + \frac{1}{2} \int_{12345} K_{123} G_{14} J_4 G_{25} J_5 \frac{\delta W_I}{\delta J_3} + \int_{123456} K_{123} G_{14} J_4 G_{25} G_{36} \frac{\delta W_I}{\delta G_{56}} + \int_{12345} K_{123} G_{24} G_{35} \frac{\delta^2 W_I}{\delta J_1 \delta G_{45}} \\ & + \int_{12345} K_{123} G_{24} G_{35} \frac{\delta W_I}{\delta J_1} \frac{\delta W_I}{\delta G_{45}} - 2 \int_{123456} L_{1234} G_{12} G_{35} G_{46} \frac{\delta W_I}{\delta G_{56}} - \frac{2}{3} \int_{12345678} L_{1234} G_{15} J_5 G_{26} J_6 G_{37} G_{48} \frac{\delta W_I}{\delta G_{78}} \\ & - \frac{2}{3} \int_{12345678} L_{1234} G_{15} G_{26} G_{37} G_{48} \frac{\delta^2 W_I}{\delta G_{56} \delta G_{78}} - \frac{2}{3} \int_{12345678} L_{1234} G_{15} G_{26} G_{37} G_{48} \frac{\delta W_I}{\delta G_{56}} \frac{\delta W_I}{\delta G_{78}} = 0. \quad (66) \end{aligned}$$

### C. Change of variables

Instead of representing Eqs. (64)–(66) graphically, let us first perform a change of variables that reduces the amount of work needed for solving the recursion relations to be derived. Since  $J$  is always connected to a free propagator  $G$ , we can as well define a modified current

$$\bar{J}_1 = \int_2 G_{12} J_2, \quad (67)$$

which already incorporates this propagator. Then Eq. (64) can be rewritten as

$$W_0[C, \bar{J}, G] = -C + \frac{1}{2} \int_{12} G_{12}^{-1} \bar{J}_1 \bar{J}_2 - \frac{1}{2} \int_1 (\ln G^{-1})_{11}. \quad (68)$$

Performing this change of variables introduces into the  $W$  identities double derivatives with respect to  $\bar{J}$  which we would like to avoid in favor of derivatives with respect to free correlation function  $G$ . This can be achieved with the result

$$\left(\frac{\delta^2 W_I}{\delta \bar{J}_1 \delta \bar{J}_2}\right)_G + \left(\frac{\delta W_I}{\delta \bar{J}_1}\right)_G \left(\frac{\delta W_I}{\delta \bar{J}_2}\right)_G = 2 \left(\frac{\delta W_I}{\delta G_{12}}\right)_J \quad (69)$$

of Appendix B.

Equations (65) and (66) then become

$$\begin{aligned} & \int_1 \bar{J}_1 \frac{\delta W_I}{\delta \bar{J}_1} - \frac{1}{2} \int_{123} K_{123} G_{12} \bar{J}_3 - \frac{1}{2} \int_{123} K_{123} \bar{J}_1 \bar{J}_2 \bar{J}_3 + \frac{1}{2} \int_{1234} L_{1234} G_{12} \bar{J}_3 \bar{J}_4 + \frac{1}{6} \int_{1234} L_{1234} \bar{J}_1 \bar{J}_2 \bar{J}_3 \bar{J}_4 - \int_{12345} K_{123} \bar{J}_1 G_{24} G_{35} \frac{\delta W_I}{\delta G_{45}} \\ & - \int_{1234} K_{123} \bar{J}_1 \bar{J}_2 G_{34} \frac{\delta W_I}{\delta \bar{J}_4} + \frac{1}{2} \int_{12345} L_{1234} G_{12} \bar{J}_3 G_{45} \frac{\delta W_I}{\delta \bar{J}_5} + \frac{1}{2} \int_{12345} L_{1234} \bar{J}_1 \bar{J}_2 \bar{J}_3 G_{45} \frac{\delta W_I}{\delta \bar{J}_5} + \int_{123456} L_{1234} \bar{J}_1 \bar{J}_2 G_{35} G_{46} \frac{\delta W_I}{\delta G_{56}} \\ & + \frac{1}{3} \int_{1234567} L_{1234} \bar{J}_1 G_{25} G_{36} G_{47} \frac{\delta^2 W_I}{\delta \bar{J}_5 \delta G_{67}} + \frac{1}{3} \int_{1234567} L_{1234} \bar{J}_1 G_{25} G_{36} G_{47} \frac{\delta W_I}{\delta \bar{J}_5} \frac{\delta W_I}{\delta G_{67}} = 0 \end{aligned} \quad (70)$$

and

$$\begin{aligned} & 2 \int_{12} G_{12} \frac{\delta W_I}{\delta G_{12}} + \int_1 \bar{J}_1 \frac{\delta W_I}{\delta \bar{J}_1} - \frac{3}{2} \int_{123} K_{123} G_{12} \bar{J}_3 - \frac{1}{2} \int_{123} K_{123} \bar{J}_1 \bar{J}_2 \bar{J}_3 + \frac{1}{2} \int_{1234} L_{1234} G_{12} G_{34} + \int_{1234} L_{1234} G_{12} \bar{J}_3 \bar{J}_4 \\ & + \frac{1}{6} \int_{1234} L_{1234} \bar{J}_1 \bar{J}_2 \bar{J}_3 \bar{J}_4 - \frac{3}{2} \int_{1234} K_{123} G_{12} G_{34} \frac{\delta W_I}{\delta \bar{J}_4} - \frac{3}{2} \int_{1234} K_{123} \bar{J}_1 \bar{J}_2 G_{34} \frac{\delta W_I}{\delta \bar{J}_4} - 3 \int_{12345} K_{123} \bar{J}_1 G_{24} G_{35} \frac{\delta W_I}{\delta G_{45}} \\ & - \int_{123456} K_{123} G_{14} G_{25} G_{36} \frac{\delta^2 W_I}{\delta \bar{J}_4 \delta G_{56}} - \int_{123456} K_{123} G_{24} G_{35} G_{16} \frac{\delta W_I}{\delta \bar{J}_6} \frac{\delta W_I}{\delta G_{45}} + 2 \int_{123456} L_{1234} G_{12} G_{35} G_{46} \frac{\delta W_I}{\delta G_{56}} \\ & + 2 \int_{12345} L_{1234} G_{12} \bar{J}_3 G_{45} \frac{\delta W_I}{\delta \bar{J}_5} + 2 \int_{123456} L_{1234} \bar{J}_1 \bar{J}_2 G_{35} G_{46} \frac{\delta W_I}{\delta G_{56}} + \frac{2}{3} \int_{12345} L_{1234} \bar{J}_1 \bar{J}_2 \bar{J}_3 G_{45} \frac{\delta W_I}{\delta \bar{J}_5} \\ & + \frac{2}{3} \int_{12345678} L_{1234} G_{15} G_{26} G_{37} G_{48} \frac{\delta^2 W_I}{\delta G_{56} \delta G_{78}} + \frac{4}{3} \int_{1234567} L_{1234} \bar{J}_1 G_{25} G_{36} G_{47} \frac{\delta^2 W_I}{\delta G_{56} \delta \bar{J}_7} \\ & + \frac{2}{3} \int_{12345678} L_{1234} G_{15} G_{26} G_{37} G_{48} \frac{\delta W_I}{\delta G_{56}} \frac{\delta W_I}{\delta G_{78}} + \frac{4}{3} \int_{1234567} L_{1234} \bar{J}_1 G_{25} G_{36} G_{47} \frac{\delta W_I}{\delta \bar{J}_5} \frac{\delta W_I}{\delta G_{67}} = 0. \end{aligned} \quad (71)$$

To represent Eqs. (70) and (71) graphically, write for the derivatives of  $W_I$  with respect to  $\bar{J}$  and  $G$

$$-\frac{\delta W_I}{\delta \bar{J}_1} = 1 \circlearrowleft (W_I), \quad \frac{\delta W_I}{\delta G_{12}} = \begin{array}{c} 1 \\ \bullet \\ \circlearrowleft (W_I) \\ \bullet \\ 2 \end{array}, \quad -\frac{\delta^2 W_I}{\delta \bar{J}_3 \delta G_{12}} = \begin{array}{c} 2 \\ \bullet \\ \bullet \\ \circlearrowleft (W_I) \\ \bullet \\ 3 \end{array}, \quad \frac{\delta^2 W_I}{\delta G_{12} \delta G_{34}} = \begin{array}{c} 2 \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \circlearrowleft (W_I) \\ \bullet \\ 3 \\ \bullet \\ 4 \end{array} \quad (72)$$

and use the vertices

$$-L_{1234} = \begin{array}{c} 2 \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ 3 \end{array} \begin{array}{c} 1 \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ 4 \end{array}, \quad -K_{123} = \begin{array}{c} 1 \\ \bullet \\ \bullet \\ \bullet \\ 2 \end{array} \begin{array}{c} 3 \\ \bullet \\ \bullet \\ \bullet \\ 3 \end{array}, \quad -\bar{J}_1 = \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ 1 \end{array}, \quad -C = \bullet. \quad (73)$$

Propagators  $G$  are indicated by lines connected at both ends. The double line on  $\bar{J}$  indicates that the propagator is absorbed into our new current  $\bar{J}$ , so derivatives with respect to  $G$  act only on propagators not connected to a current, i.e., on single lines [see, however, Eq. (75)]. All space arguments that are not indicated by numbers are integrated over.

We can write Eq. (68) now as

$$W_0[C, \bar{J}, G] = \bullet + \frac{1}{2} \bullet\bullet + \frac{1}{2} \circlearrowleft, \quad (74)$$

where by definition

$$\bullet\text{---}\bullet = \int_{12} G_{12}^{-1} \bar{J}_1 \bar{J}_2, \tag{75}$$

Eq. (70) as

$$\begin{aligned} & \bullet\text{---}\bigcirc W_I \\ &= \frac{1}{2} \text{Y-vertex} + \frac{1}{6} \text{X-vertex} + \frac{1}{2} \text{circle with dot} + \frac{1}{2} \text{circle with two dots} \\ &+ \text{Y-vertex} \bigcirc W_I + \frac{1}{2} \text{X-vertex} \bigcirc W_I + \frac{1}{2} \text{circle with dot} \bigcirc W_I + \text{circle with two dots} \bigcirc W_I \\ &+ \frac{1}{3} \text{circle with two dots} \bigcirc W_I + \frac{1}{3} \text{circle with dot} \bigcirc W_I \end{aligned} \tag{76}$$

and Eq. (71) as

$$\begin{aligned} & 2 \text{circle with two dots} + \bullet\text{---}\bigcirc W_I \\ &= \frac{1}{2} \text{Y-vertex} + \frac{1}{6} \text{X-vertex} + \frac{3}{2} \text{circle with dot} + \text{circle with two dots} + \frac{1}{2} \text{two circles} \\ &+ \frac{3}{2} \text{Y-vertex} \bigcirc W_I + \frac{2}{3} \text{X-vertex} \bigcirc W_I + \frac{3}{2} \text{circle with dot} \bigcirc W_I + 2 \text{circle with dot} \bigcirc W_I + 3 \text{circle with two dots} \bigcirc W_I \\ &+ 2 \text{circle with two dots} \bigcirc W_I + 2 \text{two circles} \bigcirc W_I + \text{circle with two dots} \bigcirc W_I + \frac{4}{3} \text{circle with dot} \bigcirc W_I + \frac{2}{3} \text{circle with two dots} \bigcirc W_I \\ &+ \text{circle with dot} \bigcirc W_I + \frac{4}{3} \text{circle with dot} \bigcirc W_I + \frac{2}{3} \text{two circles} \bigcirc W_I \end{aligned} \tag{77}$$

By construction, the simpler equation (76) involves only the  $J$ -dependent terms and is therefore by itself not sufficient for an investigation of the  $J$ -independent terms, for which Eq. (77) has to be used.

**D. Recursion relations**

For later use note the following topological relations. Let  $n_4$  be the number of four-vertices,  $n_3$  the number of three-vertices,  $n_1$  the number of  $\bar{J}_s$ ,  $n_G$  the number of free propagators  $G$  not connected to a  $J$  and  $L$  the number of loops in a connected diagram  $D$ . Then

$$3n_3 + 4n_4 = n_1 + 2n_G, \quad n_3 + 2n_4 = 2(L - 1) + n_1, \tag{78}$$

and

$$\bullet\text{---}\bigcirc D = n_1 \bigcirc D, \quad \text{circle with two dots} \bigcirc D = n_G \bigcirc D. \tag{79}$$

It is useful to consider a double expansion in the number  $L$  of loops and powers  $n$  of  $J$  or  $\bar{J}$ ,

$$W \equiv \left( W \right) = \sum_{L=0}^{\infty} \sum_{n=0}^{\infty} W^{(L,n)} \equiv \sum_{L=0}^{\infty} \sum_{n=0}^{\infty} \left( \begin{matrix} L \\ n \end{matrix} \right) . \tag{80}$$

Then the  $L$ -loop contribution to the connected  $n$ -point function with vanishing source  $J$  is given by

$$G_{i_1, \dots, i_n}^{(c)(L,n)} = \frac{\delta^n}{\delta J_{i_1} \dots \delta J_{i_n}} W^{(L)}|_{J=0} = \frac{\delta^n}{\delta J_{i_1} \dots \delta J_{i_n}} W^{(L,n)} . \tag{81}$$

We have

$$\left( \begin{matrix} 0 \\ 1 \end{matrix} \right) = 0 \tag{82}$$

and from Eq. (74)

$$\left( \begin{matrix} 0 \\ 0 \end{matrix} \right) = \bullet , \quad \left( \begin{matrix} 0 \\ 2 \end{matrix} \right) = \frac{1}{2} \text{---}\bullet\text{---}\bullet , \quad \left( \begin{matrix} 1 \\ 0 \end{matrix} \right) = \frac{1}{2} \bigcirc . \tag{83}$$

The other  $W^{(L,n)}$  constitute  $W_J$ .

Using

$$\text{---}\left( \begin{matrix} L \\ n \end{matrix} \right) = n \left( \begin{matrix} L \\ n \end{matrix} \right) , \tag{84}$$

Eq. (76) can be split into

$$\left( \begin{matrix} 0 \\ 3 \end{matrix} \right) = \frac{1}{6} \text{---}\bullet\text{---}\bullet\text{---}\bullet , \tag{85}$$

$$\left( \begin{matrix} 0 \\ 4 \end{matrix} \right) = \frac{1}{24} \text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---}\bullet + \frac{1}{4} \text{---}\bullet\text{---}\bullet\text{---}\left( \begin{matrix} 0 \\ 3 \end{matrix} \right) + \frac{1}{4} \text{---}\bullet\text{---}\left( \begin{matrix} 0 \\ 3 \end{matrix} \right) = \frac{1}{24} \text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---}\bullet + \frac{1}{8} \text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---}\bullet , \tag{86}$$

$$\left( \begin{matrix} 1 \\ 1 \end{matrix} \right) = \frac{1}{2} \text{---}\bullet\text{---}\bigcirc , \tag{87}$$

$$\left( \begin{matrix} 1 \\ 2 \end{matrix} \right) = \frac{1}{4} \text{---}\bullet\text{---}\bigcirc + \frac{1}{2} \text{---}\bullet\text{---}\left( \begin{matrix} 1 \\ 1 \end{matrix} \right) + \frac{1}{2} \text{---}\bullet\text{---}\left( \begin{matrix} 1 \\ 1 \end{matrix} \right) = \frac{1}{4} \text{---}\bullet\text{---}\bigcirc + \frac{1}{4} \text{---}\bullet\text{---}\bigcirc + \frac{1}{4} \text{---}\bullet\text{---}\bigcirc\text{---}\bullet , \tag{88}$$

and the recursion relation

$$\begin{aligned} n \left( \begin{matrix} L \\ n \end{matrix} \right) &\doteq \text{---}\left( \begin{matrix} L \\ n-1 \end{matrix} \right) + \frac{1}{2} \text{---}\left( \begin{matrix} L \\ n-2 \end{matrix} \right) + \frac{1}{2} \text{---}\left( \begin{matrix} L-1 \\ n \end{matrix} \right) + \text{---}\left( \begin{matrix} L \\ n-1 \end{matrix} \right) + \text{---}\left( \begin{matrix} L \\ n-2 \end{matrix} \right) \\ &+ \frac{1}{3} \text{---}\left( \begin{matrix} L-1 \\ n \end{matrix} \right) + \frac{1}{3} \sum_{l=0}^L \sum_{m=1}^n \left( \begin{matrix} l \\ m \end{matrix} \right) \text{---}\left( \begin{matrix} L-l \\ n-m \end{matrix} \right) , \end{aligned} \tag{89}$$

where the dot on the equal sign means that the right hand side only involves  $W^{(i,j)}$  that are part of  $W_I$ , i.e., excluding  $(i,j) \in \{(0,0), (0,1), (0,2), (1,0)\}$  and negative  $i$  or  $j$ . Equation (89) is valid for all  $W^{(L,n)}$  which are part of  $W_I$  with the exception of  $(L,n) \in \{(0,3), (0,4), (1,1), (1,2)\}$ .

From Eq. (77) follow again equations leading with Eq. (79) to Eqs. (85)–(88), but also

$$\begin{aligned}
 2 \begin{array}{c} \circlearrowleft \\ \bullet \\ 2 \\ \bullet \\ 0 \end{array} &= \frac{1}{2} \begin{array}{c} \circlearrowleft \\ \bullet \\ \bullet \\ \circlearrowleft \end{array} + \frac{3}{2} \begin{array}{c} \circlearrowleft \\ \bullet \\ \circlearrowleft \\ \bullet \\ \bullet \\ \circlearrowleft \\ \bullet \\ 1 \\ \bullet \\ 1 \end{array} + \begin{array}{c} \circlearrowleft \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ 1 \\ \bullet \\ 1 \end{array} \\
 &= \frac{1}{2} \begin{array}{c} \circlearrowleft \\ \bullet \\ \bullet \\ \circlearrowleft \end{array} + \frac{3}{4} \begin{array}{c} \circlearrowleft \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \circlearrowleft \end{array} + \frac{1}{2} \begin{array}{c} \circlearrowleft \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array}, \tag{90}
 \end{aligned}$$

which with Eq. (79) becomes

$$\begin{array}{c} \circlearrowleft \\ \bullet \\ 2 \\ \bullet \\ 0 \end{array} = \frac{1}{8} \begin{array}{c} \circlearrowleft \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \circlearrowleft \end{array} + \frac{1}{12} \begin{array}{c} \circlearrowleft \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} + \frac{1}{8} \begin{array}{c} \circlearrowleft \\ \bullet \\ \bullet \\ \circlearrowleft \end{array}, \tag{91}$$

and a recursion relation which we write down only for  $n=0$ , since for  $n>0$  the simpler relation (89) can be used:

$$\begin{aligned}
 \begin{array}{c} \circlearrowleft \\ \bullet \\ L \\ \bullet \\ 0 \end{array} &= \frac{3}{4} \begin{array}{c} \circlearrowleft \\ \bullet \\ L-1 \\ \bullet \\ 1 \end{array} + \begin{array}{c} \circlearrowleft \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ L-1 \\ \bullet \\ 0 \end{array} + \frac{1}{2} \begin{array}{c} \circlearrowleft \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ L-1 \\ \bullet \\ 1 \end{array} + \frac{1}{3} \begin{array}{c} \circlearrowleft \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ L-1 \\ \bullet \\ 0 \end{array} \\
 &+ \frac{1}{2} \sum_{l=1}^{L-2} \begin{array}{c} \circlearrowleft \\ \bullet \\ l \\ \bullet \\ 1 \end{array} \begin{array}{c} \circlearrowleft \\ \bullet \\ L-l \\ \bullet \\ 0 \end{array} + \frac{1}{3} \sum_{l=2}^{L-2} \begin{array}{c} \circlearrowleft \\ \bullet \\ l \\ \bullet \\ 0 \end{array} \begin{array}{c} \circlearrowleft \\ \bullet \\ L-l \\ \bullet \\ 0 \end{array}. \tag{92}
 \end{aligned}$$

Equation (92) is valid for  $L>2$ .

Note that in Eq. (92), but not in Eq. (89), the right-hand side involves graphs with more legs than the left-hand side, namely, one more. This implies that for the generation of vacuum graphs, we have to consider also one-point functions. For all others it is enough to consider only diagrams with equal or less numbers of legs. Note further that if all lower loop orders contain only connected graphs, then the recursion relations generate only connected graphs. This establishes by induction that  $W$  generates only connected graphs, as shown before in Refs. [1,2].

As an example, we compute  $W^{(3,0)}$  in Appendix C. Combining Eqs. (83), (91), and the result (C13) of Appendix C, we get  $W$  at  $J=0$  in the three-loop approximation

$$\begin{aligned}
 W[J=0] &= \bullet + \frac{1}{2} \begin{array}{c} \circlearrowleft \end{array} + \frac{1}{8} \begin{array}{c} \circlearrowleft \\ \bullet \\ \bullet \\ \circlearrowleft \end{array} + \frac{1}{12} \begin{array}{c} \circlearrowleft \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \circlearrowleft \end{array} + \frac{1}{8} \begin{array}{c} \circlearrowleft \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \\
 &+ \frac{1}{16} \begin{array}{c} \circlearrowleft \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} + \frac{1}{48} \begin{array}{c} \circlearrowleft \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} + \frac{1}{8} \begin{array}{c} \circlearrowleft \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} + \frac{1}{16} \begin{array}{c} \circlearrowleft \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} + \frac{1}{24} \begin{array}{c} \circlearrowleft \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \\
 &+ \frac{1}{16} \begin{array}{c} \circlearrowleft \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} + \frac{1}{8} \begin{array}{c} \circlearrowleft \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} + \frac{1}{12} \begin{array}{c} \circlearrowleft \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} + \frac{1}{8} \begin{array}{c} \circlearrowleft \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} + \frac{1}{8} \begin{array}{c} \circlearrowleft \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \\
 &+ \frac{1}{16} \begin{array}{c} \circlearrowleft \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} + \frac{1}{48} \begin{array}{c} \circlearrowleft \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array}. \tag{93}
 \end{aligned}$$

At this low loop order, it is still relatively easy to check that the weights come out the same when using the combinatorial prescriptions that come with the usual Feynman rules. I have written a computer code implementing the recursion relations for the connected graphs. If we restrict ourselves to the symmetric case, it reproduces the graphs and multiplicities (trivially related to the weights, see Ref. [3]) of Tables I–III in Ref. [3] and also all relevant entries in Tables V through VII there.

Except for the vacuum diagrams, we still have to use Eq. (81) to convert the graphs representing  $W$  into connected Green's functions. For example, to compute the zero-loop contribution  $G_{1234}^{(c)(0,4)}$  to the connected four-point function we combine Eqs.

(81)  $\rightarrow$  (82)  $\rightarrow$  (83)

$$G_{1234}^{(c)(0,4)} = \frac{\delta^4}{\delta J_1 \delta J_2 \delta J_3 \delta J_4} \left[ \frac{1}{24} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} + \frac{1}{8} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} \right] = \begin{array}{c} 1 \quad 3 \\ \diagdown \quad \diagup \\ 2 \quad 4 \end{array} + \begin{array}{c} 1 \quad 3 \\ \diagdown \quad \bullet \quad \diagup \\ 2 \quad 4 \end{array} + \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \bullet \quad \diagup \\ 3 \quad 4 \end{array} + \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \bullet \quad \diagup \\ 4 \quad 3 \end{array}. \quad (94)$$

That is, each diagram with  $n$  external currents is multiplied by  $n!$ , supplied by external arguments replacing the  $J$ s and then splits into ‘‘crossed’’ graphs related by exchanging external arguments. The external legs represent free correlation functions  $G$ .

#### IV. EFFECTIVE ENERGY

Often in field theory, one is interested rather in the effective energy or effective action  $\Gamma$  than the free energy  $W$  and rather in the 1PI Feynman diagrams than the connected ones. Therefore, we translate in the following the identities for  $W$  into identities for  $\Gamma$  and derive recursion relations for the 1PI Feynman diagrams representing the proper vertices.

##### A. Relations between $W$ and $\Gamma$

Since the physical situation in which we are interested does not necessarily correspond to  $J=0$ , let us for the purpose of performing a Legendre transform introduce an additional source  $\hat{J}$  into the definition of the partition function  $Z$  and the negative free energy  $W$ ,

$$\begin{aligned} Z[\hat{J}, C, J, G, K, L] &= \exp[W[\hat{J}, C, J, G, K, L]] \\ &= \int D\phi \exp\left(-E[\phi, C, J, G, K, L] \right. \\ &\quad \left. + \int_1 \hat{J}_1 \phi_1\right). \end{aligned} \quad (95)$$

Note that we trivially have the relations

$$Z[\hat{J}, C, J, G, K, L] = Z[C, J - \hat{J}, G, K, L] \quad (96)$$

and

$$W[\hat{J}, C, J, G, K, L] = W[C, J - \hat{J}, G, K, L] \quad (97)$$

between the quantities defined in Eq. (54) and those in Eq. (95). With Eq. (64) it follows then that

$$\begin{aligned} W_0[\hat{J}, C, J, G] &\equiv W[\hat{J}, C, J, G, K, L]_{K,L=0} \\ &= W[C, J - \hat{J}, G, K, L]_{K,L=0} \\ &= W_0[C, J, -\hat{J}, G] \\ &= -C + \frac{1}{2} \int_{12} G_{12} (J_1 - \hat{J}_1) (J_2 - \hat{J}_2) \\ &\quad - \frac{1}{2} \int_1 (\ln G^{-1})_{11}. \end{aligned} \quad (98)$$

Define the effective energy  $\Gamma$  by the Legendre transform

$$\Gamma[\Phi, C, J, G, K, L] = -W[\hat{J}, C, J, G, K, L] + \int_1 \hat{J}_1 \Phi_1, \quad (99)$$

where the new variable  $\Phi$  is defined by

$$\Phi_1 = \left( \frac{\delta W}{\delta \hat{J}_1} \right)_{C, J, G, K, L}, \quad (100)$$

which implicitly defines  $\hat{J}$  as functional of  $\Phi$ . As usual we have

$$\left( \frac{\delta \Gamma}{\delta \Phi_1} \right)_{C, J, G, K, L} = \hat{J}_1. \quad (101)$$

Notice that as intended by introducing the extra source term and performing the Legendre transform with respect to  $\hat{J}$  instead of  $J$ , we do not have to set  $J=0$  but only  $\hat{J}=0$  to have a proper effective energy giving us the equation of state (or the equation of motion if we consider an effective action) through its stationary points.

In the following, we assume  $C, J, K$ , and  $L$  to be fixed and do not treat them as variables. Let us use the notation

$$\left( \frac{\delta^2 F}{\delta x_1 \delta x_2} \right)_{(y_1, y_2)} \equiv \left( \frac{\delta}{\delta x_1} \right)_{y_1} \left( \frac{\delta}{\delta x_2} \right)_{y_2} F. \quad (102)$$

For deriving identities for  $\Gamma$  we first need some relations between the functional derivatives of  $W$  and  $\Gamma$ . With Eqs. (100) and (101) we get

$$P_{12} \equiv \left( \frac{\delta^2 W}{\delta \hat{J}_1 \delta \hat{J}_2} \right)_G = \left( \frac{\delta \Phi_2}{\delta \hat{J}_1} \right)_G = \left( \frac{\delta \hat{J}_1}{\delta \Phi_2} \right)_G^{-1} = \left( \frac{\delta^2 \Gamma}{\delta \Phi_1 \delta \Phi_2} \right)_G^{-1}. \quad (103)$$

For  $\Phi=0$ ,  $P$  is the usual propagator. It will turn out useful to reexpress  $P_{12}$  as

$$\begin{aligned} P_{12} &= \left( \frac{\delta^2 \ln Z}{\delta \hat{J}_1 \delta \hat{J}_2} \right)_G \\ &= \frac{1}{Z} \left( \frac{\delta^2 Z}{\delta \hat{J}_1 \delta \hat{J}_2} \right)_G - \frac{1}{Z^2} \left( \frac{\delta Z}{\delta \hat{J}_1} \right)_G \left( \frac{\delta Z}{\delta \hat{J}_2} \right)_G \\ &= -\frac{2}{Z} \left( \frac{\delta Z}{\delta G_{12}^{-1}} \right)_j - \frac{1}{Z^2} \left( \frac{\delta Z}{\delta \hat{J}_1} \right)_G \left( \frac{\delta Z}{\delta \hat{J}_2} \right)_G \\ &= -2 \left( \frac{\delta W}{\delta G_{12}^{-1}} \right)_j - \left( \frac{\delta W}{\delta \hat{J}_1} \right)_G \left( \frac{\delta W}{\delta \hat{J}_2} \right)_G \\ &= 2 \left( \frac{\delta \Gamma}{\delta G_{12}^{-1}} \right)_\Phi - \Phi_1 \Phi_2, \end{aligned} \quad (104)$$

where we have used

$$\begin{aligned}
\left(\frac{\delta W}{\delta G_{12}}\right)_j &= \left(\frac{\delta}{\delta G_{12}}\right)_j \left[ \int_3 \hat{j}_3 \Phi_3 - \Gamma \right] \\
&= \int_3 \hat{j}_3 \left(\frac{\delta \Phi_3}{\delta G_{12}}\right)_j - \left(\frac{\delta \Gamma}{\delta G_{12}}\right)_j \\
&= \int_3 \hat{j}_3 \left(\frac{\delta \Phi_3}{\delta G_{12}}\right)_j - \left[ \left(\frac{\delta \Gamma}{\delta G_{12}}\right)_\Phi \right. \\
&\quad \left. + \int_3 \left(\frac{\delta \Gamma}{\delta \Phi_3}\right)_G \left(\frac{\delta \Phi_3}{\delta G_{12}}\right)_j \right] \\
&= - \left(\frac{\delta \Gamma}{\delta G_{12}}\right)_\Phi. \tag{105}
\end{aligned}$$

Further we have

$$\begin{aligned}
\left(\frac{\delta^2 W}{\delta \hat{J}_1 \delta G_{23}}\right)_{(G\hat{J})} &= \left(\frac{\delta \Phi_1}{\delta G_{23}}\right)_j = - \int_4 \left(\frac{\delta \Phi_1}{\delta \hat{J}_4}\right)_G \left(\frac{\delta \hat{J}_4}{\delta G_{23}}\right)_\Phi \\
&= - \int_4 P_{14} \left(\frac{\delta^2 \Gamma}{\delta \Phi_4 \delta G_{23}}\right)_{(G\Phi)} \tag{106}
\end{aligned}$$

and

$$\begin{aligned}
\left(\frac{\delta^2 W}{\delta G_{12} \delta G_{34}}\right)_j &= - \left(\frac{\delta^2 \Gamma}{\delta G_{12} \delta G_{34}}\right)_{(\hat{J}\Phi)} \\
&= - \left(\frac{\delta^2 \Gamma}{\delta G_{12} \delta G_{34}}\right)_\Phi - \int_5 \left(\frac{\delta^2 \Gamma}{\delta \Phi_5 \delta G_{34}}\right)_{(G\Phi)} \\
&\quad \times \left(\frac{\delta \Phi_5}{\delta G_{12}}\right)_j \\
&= - \left(\frac{\delta^2 \Gamma}{\delta G_{12} \delta G_{34}}\right)_\Phi \\
&\quad + \int_{56} \left(\frac{\delta^2 \Gamma}{\delta \Phi_5 \delta G_{34}}\right)_{(G\Phi)} P_{56} \left(\frac{\delta^2 \Gamma}{\delta \Phi_6 \delta G_{12}}\right)_{(G\Phi)}. \tag{107}
\end{aligned}$$

### B. Identities for $\Gamma_I$

Making use of Eq. (97) and the relations just derived, Eqs. (55) and (56) can be rewritten as

$$\begin{aligned}
0 &= -J_1 + \frac{\delta \Gamma}{\delta \Phi_1} - \int_2 G_{12}^{-1} \Phi_2 - \int_{23} K_{123} \frac{\delta \Gamma}{\delta G_{23}^{-1}} \\
&\quad - \frac{1}{3} \int_{234} L_{1234} \left( \int_5 P_{25} \frac{\delta^2 \Gamma}{\delta \Phi_5 \delta G_{34}^{-1}} + \Phi_2 \frac{\delta \Gamma}{\delta G_{34}^{-1}} \right) \tag{108}
\end{aligned}$$

and

$$\begin{aligned}
0 &= \delta_{12} - J_1 \Phi_2 + \frac{\delta \Gamma}{\delta \Phi_1} \Phi_2 - 2 \int_3 G_{13}^{-1} \frac{\delta \Gamma}{\delta G_{23}^{-1}} \\
&\quad - \int_{34} K_{134} \left( \int_5 P_{25} \frac{\delta^2 \Gamma}{\delta \Phi_5 \delta G_{34}^{-1}} + \Phi_2 \frac{\delta \Gamma}{\delta G_{34}^{-1}} \right) \\
&\quad + \frac{2}{3} \int_{345} L_{1345} \left( \frac{\delta^2 \Gamma}{\delta G_{23}^{-1} \delta G_{45}^{-1}} \right. \\
&\quad \left. - \int_{67} \frac{\delta^2 \Gamma}{\delta \Phi_6 \delta G_{45}^{-1}} P_{67} \frac{\delta^2 \Gamma}{\delta \Phi_7 \delta G_{23}^{-1}} - \frac{\delta \Gamma}{\delta G_{23}^{-1}} \frac{\delta \Gamma}{\delta G_{45}^{-1}} \right). \tag{109}
\end{aligned}$$

We have omitted now indicating the variables that are kept fixed, since everything is written in terms of the variables  $\Phi$  and  $G$ .

Split  $\Gamma$  into a free and an interacting part

$$\Gamma = \Gamma_0 + \Gamma_I \equiv \Gamma|_{K,L=0} + \Gamma_I. \tag{110}$$

Then,

$$\begin{aligned}
\Gamma_0[\Phi, C, J, G] &= -W_0[\hat{J}, C, J, G] + \int_1 \hat{J}_1 \Phi_1 \\
&= -W_0[C, J - \hat{J}, G] + \int_1 \hat{J}_1 \Phi_1 \tag{111}
\end{aligned}$$

with

$$\Phi_1 = \frac{\delta W_0[\hat{J}, C, J, G]}{\delta \hat{J}_1} = \frac{\delta W_0[C, J - \hat{J}, G]}{\delta \hat{J}_1} = \int_2 G_{12}(\hat{J}_2 - J_2), \tag{112}$$

i.e.,

$$\hat{J}_1 = J_1 + \int_2 G_{12}^{-1} \Phi_2. \tag{113}$$

Using Eq. (64), we get

$$\begin{aligned}
\Gamma_0[\Phi, C, J, G] &= C + \frac{1}{2} \int_1 (\ln G^{-1})_{11} + \int_1 J_1 \Phi_1 \\
&\quad + \frac{1}{2} \int_{12} G_{12}^{-1} \Phi_1 \Phi_2. \tag{114}
\end{aligned}$$

For  $\Gamma_0$ , Eqs. (108) and (109) reduce to

$$0 = -J_1 + \frac{\delta \Gamma_0}{\delta \Phi_1} - \int_2 G_{12}^{-1} \Phi_2 \tag{115}$$

and

$$0 = \delta_{12} - J_1 \Phi_2 + \frac{\delta \Gamma_0}{\delta \Phi_1} \Phi_2 - 2 \int_3 G_{13}^{-1} \frac{\delta \Gamma_0}{\delta G_{23}^{-1}}, \tag{116}$$

respectively, so that

$$\frac{\delta\Gamma_0}{\delta\Phi_1} = J_1 + \int_2 G_{12}^{-1} \Phi_2 \quad \text{and} \quad (117)$$

and

$$\frac{\delta^2\Gamma_0}{\delta G_{12}^{-1} \delta G_{34}^{-1}} = -\frac{1}{4}(G_{13}G_{24} + G_{14}G_{23}). \quad (121)$$

$$\frac{\delta\Gamma_0}{\delta G_{12}^{-1}} = \frac{1}{2}(G_{12} + \Phi_1\Phi_2). \quad (118)$$

Notice that with Eqs. (104), (118), and (A12) we get

In the following we also need

$$\frac{\delta^2\Gamma_0}{\delta\Phi_1 \delta\Phi_2} = G_{12}^{-1}, \quad (119)$$

$$P_{12} = G_{12} + 2 \frac{\delta\Gamma_I}{\delta G_{12}^{-1}} = G_{12} - 2 \int_{34} G_{13}G_{24} \frac{\delta\Gamma_I}{\delta G_{34}}. \quad (122)$$

$$\frac{\delta^2\Gamma_0}{\delta\Phi_1 \delta G_{23}^{-1}} = \frac{1}{2}(\delta_{12}\Phi_3 + \delta_{13}\Phi_2), \quad (120)$$

Subtracting Eq. (115) from Eq. (108), multiplying with  $\Phi_1$ , integrating over  $x_1$  and using Eqs. (118), (120), and (A12) gives

$$\begin{aligned} 0 = & \int_1 \frac{\delta\Gamma_I}{\delta\Phi_1} \Phi_1 - \frac{1}{2} \int_{123} K_{123} \Phi_1 \Phi_2 \Phi_3 - \frac{1}{6} \int_{1234} L_{1234} \Phi_1 \Phi_2 \Phi_3 \Phi_4 - \frac{1}{2} \int_{123} K_{123} \Phi_1 G_{23} - \frac{1}{2} \int_{1234} L_{1234} G_{12} \Phi_3 \Phi_4 \\ & + \int_{12345} K_{123} \Phi_1 G_{24} G_{35} \frac{\delta\Gamma_I}{\delta G_{45}} + \int_{123456} L_{1234} \Phi_1 \Phi_2 G_{35} G_{46} \frac{\delta\Gamma_I}{\delta G_{56}} + \frac{1}{3} \int_{1234567} L_{1234} \Phi_1 G_{25} G_{36} G_{47} \frac{\delta^2\Gamma_I}{\delta\Phi_5 \delta G_{67}} \\ & - \frac{2}{3} \int_{123456789} L_{1234} \Phi_1 G_{25} G_{67} G_{38} G_{49} \frac{\delta\Gamma_I}{\delta G_{56}} \frac{\delta^2\Gamma_I}{\delta\Phi_7 \delta G_{89}}. \end{aligned} \quad (123)$$

Subtracting Eq. (116) from Eq. (109), setting  $x_2 = x_1$ , integrating over  $x_1$  and using Eqs. (118), (120), (121), and (A12) gives

$$\begin{aligned} 0 = & \int_1 \frac{\delta\Gamma_I}{\delta\Phi_1} \Phi_1 + 2 \int_{12} G_{12} \frac{\delta\Gamma_I}{\delta G_{12}} - \frac{1}{2} \int_{123} K_{123} \Phi_1 \Phi_2 \Phi_3 - \frac{1}{6} \int_{1234} L_{1234} \Phi_1 \Phi_2 \Phi_3 \Phi_4 - \frac{3}{2} \int_{123} K_{123} G_{12} \Phi_3 - \int_{1234} L_{1234} G_{12} \Phi_3 \Phi_4 \\ & - \frac{1}{2} \int_{1234} L_{1234} G_{12} G_{34} + 3 \int_{12345} K_{123} \Phi_1 G_{24} G_{35} \frac{\delta\Gamma_I}{\delta G_{45}} + \int_{123456} K_{123} G_{14} G_{25} G_{36} \frac{\delta\Gamma_I}{\delta\Phi_4 \delta G_{56}} \\ & - 2 \int_{12345678} K_{123} G_{34} G_{56} G_{27} G_{38} \frac{\delta\Gamma_I}{\delta G_{45}} \frac{\delta^2\Gamma_I}{\delta\Phi_6 \delta G_{78}} + 2 \int_{123456} L_{1234} \Phi_1 \Phi_2 G_{35} G_{46} \frac{\delta\Gamma_I}{\delta G_{56}} + 2 \int_{123456} L_{1234} G_{12} G_{35} G_{46} \frac{\delta\Gamma_I}{\delta G_{56}} \\ & + \frac{4}{3} \int_{1234567} L_{1234} \Phi_1 G_{25} G_{36} G_{47} \frac{\delta^2\Gamma_I}{\delta\Phi_5 \delta G_{67}} + \frac{2}{3} \int_{12345678} L_{1234} G_{15} G_{26} G_{37} G_{48} \frac{\delta^2\Gamma_I}{\delta G_{56} \delta G_{78}} \\ & - \frac{2}{3} \int_{12345678} L_{1234} G_{15} G_{26} G_{37} G_{48} \frac{\delta\Gamma_I}{\delta G_{55}} \frac{\delta\Gamma_I}{\delta G_{78}} - \frac{8}{3} \int_{123456789} L_{1234} \Phi_1 G_{25} G_{38} G_{49} G_{67} \frac{\delta\Gamma_I}{\delta G_{56}} \frac{\delta^2\Gamma_I}{\delta\Phi_7 \delta G_{89}} \\ & - \frac{2}{3} \int_{11\bar{2}\bar{2}\bar{3}\bar{3}\bar{4}\bar{4}\bar{5}\bar{6}} L_{1234} G_{11} \bar{G}_{22} \bar{G}_{33} \bar{G}_{44} \bar{G}_{56} \frac{\delta^2\Gamma_I}{\delta\Phi_5 \delta G_{12}^-} \frac{\delta^2\Gamma_I}{\delta\Phi_6 \delta G_{34}^-} \\ & + \frac{4}{3} \int_{11\bar{1}\bar{2}\bar{2}\bar{3}\bar{3}\bar{4}\bar{4}\bar{5}\bar{6}\bar{7}\bar{8}} L_{1234} G_{11} \bar{G}_{22} \bar{G}_{33} \bar{G}_{44} \bar{G}_{56} G_{78} \frac{\delta^2\Gamma_I}{\delta\Phi_5 \delta G_{12}^-} \frac{\delta\Gamma_I}{\delta G_{67}} \frac{\delta^2\Gamma_I}{\delta\Phi_8 \delta G_{34}^-}. \end{aligned} \quad (124)$$

To represent Eqs. (123) and (124) graphically, write for the derivatives of  $\Gamma_I$  with respect to  $\Phi$  and  $G$

$$-\frac{\delta\Gamma_I}{\delta\Phi_1} = 1 \circlearrowleft (\Gamma_I), \quad -\frac{\delta\Gamma_I}{\delta G_{12}} = \begin{matrix} 1 \\ \bullet \\ \circlearrowleft (\Gamma_I) \\ \bullet \\ 2 \end{matrix}, \quad -\frac{\delta^2\Gamma_I}{\delta\Phi_3 \delta G_{12}} = \begin{matrix} 2 \\ \bullet \\ \bullet \\ \circlearrowleft (\Gamma_I) \\ \bullet \\ 3 \end{matrix}, \quad -\frac{\delta^2\Gamma_I}{\delta G_{12} \delta G_{34}} = \begin{matrix} 2 \\ \bullet \\ \bullet \\ \circlearrowleft (\Gamma_I) \\ \bullet \\ 3 \\ \bullet \\ 4 \end{matrix} \quad (125)$$



and use the vertices

$$-L_{1234} = \begin{array}{c} 1 \\ \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \\ 2 \quad 3 \\ 4 \end{array}, \quad -K_{123} = \begin{array}{c} 1 \\ | \\ \bullet \\ / \quad \backslash \\ 2 \quad 3 \end{array}, \quad -J_1 = \begin{array}{c} \bullet \\ || \\ 1 \end{array}, \quad -C = \bullet. \quad (126)$$

Instances of  $\Phi$  are indicated by

$$\Phi_1 = \begin{array}{c} \circ \\ || \\ 1 \end{array}. \quad (127)$$

Free propagators  $G$  are indicated by lines connected at both ends. The double lines on  $J$  and  $\Phi$  indicate that there are no propagators attached to them in the diagrams, so derivatives with respect to  $G$  act only on single lines [see, however, Eq. (129)]. All space arguments that are not indicated by numbers are integrated over.

Now Eq. (114) can be written as

$$-\Gamma_0 = \bullet + \begin{array}{c} \circ \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \circ \end{array} + \begin{array}{c} \circ \\ \circ \end{array}, \quad (128)$$

where by definition

$$\begin{array}{c} \bullet \\ \circ \end{array} = - \int_{12} \Phi_1 G_{12}^{-1} \Phi_2, \quad (129)$$

and Eq. (122) as

$$P_{12} = 1 \text{---} 2 + 2 \begin{array}{c} \circ \\ \Gamma_I \\ \bullet \bullet \\ 1 \text{---} 2 \end{array}. \quad (130)$$

We can write Eq. (123) as

$$\begin{aligned} \begin{array}{c} \bullet \\ \circ \end{array} \Gamma_I &= \frac{1}{2} \begin{array}{c} \bullet \\ \circ \end{array} + \frac{1}{6} \begin{array}{c} \bullet \\ \bullet \end{array} + \frac{1}{2} \begin{array}{c} \bullet \\ \circ \end{array} + \frac{1}{2} \begin{array}{c} \bullet \\ \bullet \end{array} \\ &+ \begin{array}{c} \bullet \\ \circ \end{array} \Gamma_I + \begin{array}{c} \bullet \\ \bullet \end{array} \Gamma_I + \frac{1}{3} \begin{array}{c} \bullet \\ \bullet \end{array} \Gamma_I + \frac{2}{3} \begin{array}{c} \bullet \\ \bullet \end{array} \Gamma_I \end{aligned} \quad (131)$$

and Eq. (124) as

$$\begin{aligned} &2 \begin{array}{c} \bullet \\ \circ \end{array} \Gamma_I + \begin{array}{c} \bullet \\ \circ \end{array} \Gamma_I \\ &= \frac{1}{2} \begin{array}{c} \bullet \\ \circ \end{array} + \frac{1}{6} \begin{array}{c} \bullet \\ \bullet \end{array} + \frac{3}{2} \begin{array}{c} \bullet \\ \circ \end{array} + \begin{array}{c} \bullet \\ \bullet \end{array} + \frac{1}{2} \begin{array}{c} \bullet \\ \bullet \end{array} \\ &+ 3 \begin{array}{c} \bullet \\ \circ \end{array} \Gamma_I + \begin{array}{c} \bullet \\ \bullet \end{array} \Gamma_I + 2 \begin{array}{c} \bullet \\ \bullet \end{array} \Gamma_I \\ &+ 2 \begin{array}{c} \bullet \\ \bullet \end{array} \Gamma_I + 2 \begin{array}{c} \bullet \\ \bullet \end{array} \Gamma_I + \frac{4}{3} \begin{array}{c} \bullet \\ \bullet \end{array} \Gamma_I + \frac{2}{3} \begin{array}{c} \bullet \\ \bullet \end{array} \Gamma_I + \frac{2}{3} \begin{array}{c} \bullet \\ \bullet \end{array} \Gamma_I \\ &+ \frac{8}{3} \begin{array}{c} \bullet \\ \bullet \end{array} \Gamma_I + \frac{2}{3} \begin{array}{c} \bullet \\ \bullet \end{array} \Gamma_I + \frac{4}{3} \begin{array}{c} \bullet \\ \bullet \end{array} \Gamma_I. \end{aligned} \quad (132)$$

Note that in the limit where  $K=0$  and  $\Phi=0$  this is identical to Eq. (18) in the limit  $\Delta=0$  if there we replace  $W_I \rightarrow \Gamma_I$ .

**C. Recursion relations**

Consider a double expansion in the number of loops  $L$  and powers  $n$  of  $\Phi$ ,

$$-\Gamma = \left( \Gamma \right) = - \sum_{L=0}^{\infty} \sum_{n=0}^{\infty} \Gamma^{(L,n)} = \sum_{L=0}^{\infty} \sum_{n=0}^{\infty} \left( \begin{matrix} L \\ n \end{matrix} \right) \tag{133}$$

[the double-indexed circles are not identical to those in Eq. (80)]. Then the  $L$ -loop contribution to the proper  $n$ -point vertex with vanishing external field is given by

$$\Gamma_{i_1, \dots, i_n}^{(L,n)} = \frac{\delta^n}{\delta\Phi_{i_1} \dots \delta\Phi_{i_n}} \Gamma^{(L)}|_{\Phi=0} = \frac{\delta^n}{\delta\Phi_{i_1} \dots \delta\Phi_{i_n}} \Gamma^{(L,n)}. \tag{134}$$

We have from Eq. (128)

$$\left( \begin{matrix} 0 \\ 0 \end{matrix} \right) = \bullet, \quad \left( \begin{matrix} 0 \\ 1 \end{matrix} \right) = \text{---} \bullet, \quad \left( \begin{matrix} 0 \\ 2 \end{matrix} \right) = \frac{1}{2} \text{---} \bullet \text{---}, \quad \left( \begin{matrix} 1 \\ 0 \end{matrix} \right) = \frac{1}{2} \left( \bullet \right), \tag{135}$$

The other  $\Gamma^{(L,n)}$  constitute  $\Gamma_I$ .

Using

$$\text{---} \left( \begin{matrix} L \\ n \end{matrix} \right) = n \left( \begin{matrix} L \\ n \end{matrix} \right), \tag{136}$$

Eq. (131) can be split into

$$\left( \begin{matrix} 0 \\ 3 \end{matrix} \right) = \frac{1}{6} \text{---} \bullet \text{---} \bullet, \quad \left( \begin{matrix} 0 \\ 4 \end{matrix} \right) = \frac{1}{24} \text{---} \bullet \text{---} \bullet \text{---} \bullet, \quad \left( \begin{matrix} 1 \\ 1 \end{matrix} \right) = \frac{1}{2} \text{---} \bullet \text{---} \left( \bullet \right), \tag{137}$$

$$\left( \begin{matrix} 1 \\ 2 \end{matrix} \right) = \frac{1}{4} \text{---} \bullet \text{---} \left( \bullet \right) + \frac{1}{2} \text{---} \bullet \text{---} \left( \begin{matrix} 1 \\ 1 \end{matrix} \right) = \frac{1}{4} \text{---} \bullet \text{---} \left( \bullet \right) + \frac{1}{4} \text{---} \bullet \text{---} \left( \bullet \right) \tag{138}$$

and the recursion relation

$$n \left( \begin{matrix} L \\ n \end{matrix} \right) \doteq \text{---} \left( \begin{matrix} L \\ n-1 \end{matrix} \right) + \text{---} \left( \begin{matrix} L \\ n-2 \end{matrix} \right) + \frac{1}{3} \text{---} \left( \begin{matrix} L-1 \\ n \end{matrix} \right) + \frac{2}{3} \sum_{l=1}^{L-1} \sum_{m=0}^{n-1} \left( \begin{matrix} l \\ m \end{matrix} \right) \text{---} \left( \begin{matrix} L-l \\ n-m \end{matrix} \right), \tag{139}$$

where the dot on the equal sign means that the right-hand side only involves  $\Gamma^{(i,j)}$  that are part of  $\Gamma_I$ , i.e., excluding  $(i,j) \in \{(0,0), (0,1), (0,2), (1,0)\}$  and negative  $i$  or  $j$ . Equation (139) is valid for all  $\Gamma^{(L,n)}$ , which are part of  $\Gamma_I$  with the exception of  $(L,n) \in \{(0,3), (0,4), (1,1), (1,2)\}$ .

Note that from Eq. (139) it follows that

$$\left( \begin{matrix} 0 \\ n \end{matrix} \right) = 0 \tag{140}$$

for  $n > 4$ .

From Eq. (132) follow again equations leading, together with Eq. (79), to Eqs. (137) and (138), but also

$$2 \left( \begin{matrix} 2 \\ 0 \end{matrix} \right) = \frac{1}{2} \left( \bullet \right) \left( \bullet \right) + \left( \begin{matrix} 1 \\ 1 \end{matrix} \right) = \frac{1}{2} \left( \bullet \right) \left( \bullet \right) + \frac{1}{2} \left( \bullet \right) \text{---} \left( \bullet \right), \tag{141}$$

which with Eq. (79) becomes

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \frac{1}{8} \text{ (two circles joined at a point)} + \frac{1}{12} \text{ (circle with a horizontal line)}, \quad (142)$$

and a recursion relation which we write down only for  $n=0$ , since for  $n>0$  the simpler relation (139) can be used:

$$\begin{aligned} \begin{pmatrix} L \\ 0 \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} L-1 \\ 1 \end{pmatrix} + \text{ (two circles joined at a point)} + \frac{1}{3} \begin{pmatrix} L-1 \\ 0 \end{pmatrix} \\ &+ \sum_{l=1}^{L-2} \begin{pmatrix} L-l \\ 0 \end{pmatrix} \begin{pmatrix} l \\ 1 \end{pmatrix} + \frac{1}{3} \sum_{l=2}^{L-2} \begin{pmatrix} l \\ 0 \end{pmatrix} \begin{pmatrix} L-l \\ 0 \end{pmatrix} \\ &+ \frac{1}{3} \sum_{l=1}^{L-2} \begin{pmatrix} l \\ 1 \end{pmatrix} \begin{pmatrix} L-l-1 \\ 1 \end{pmatrix} + \frac{2}{3} \sum_{l_1=1}^{L-3} \sum_{l_2=1}^{L-l_1-2} \begin{pmatrix} l_1 \\ 1 \end{pmatrix} \begin{pmatrix} l_2 \\ 1 \end{pmatrix} \begin{pmatrix} L-l_1-l_2 \\ 0 \end{pmatrix}. \end{aligned} \quad (143)$$

Equation (143) is valid for  $L>2$ .

Note that in Eq. (143), but not in Eq. (139), the right-hand side involves graphs with more legs than the left-hand side, namely, one more. This implies that for the generation of vacuum graphs, we have to consider also one-point functions. For all others it is enough to consider only diagrams with equal or less numbers of legs. Note further that if all lower loop orders contain only 1PI graphs then the recursion relations also generate only 1PI graphs. This establishes by induction that  $\Gamma$  generates only 1PI graphs, as shown before in Refs. [1,2].

As an example, we compute  $\Gamma^{(3,0)}$  in Appendix D. Combining Eqs. (135), (142), and the result (D3) of Appendix D, we get the effective energy  $\Gamma$  at  $\Phi=0$  in the three-loop approximation,

$$\begin{aligned} -\Gamma[\Phi=0] &= \bullet + \frac{1}{2} \text{ (circle)} + \frac{1}{8} \text{ (two circles joined at a point)} + \frac{1}{12} \text{ (circle with a horizontal line)} \\ &+ \frac{1}{16} \text{ (three circles in a row)} + \frac{1}{48} \text{ (circle with a horizontal line and a vertical line)} + \frac{1}{8} \text{ (circle with a vertical line)} \\ &+ \frac{1}{8} \text{ (circle with a horizontal line and a vertical line)} + \frac{1}{16} \text{ (circle with a horizontal line and a vertical line)} + \frac{1}{24} \text{ (circle with a horizontal line and a vertical line)}, \end{aligned} \quad (144)$$

where propagator and vertices may contain a background-field dependence as given, e.g., by Eqs. (49)–(53). The corresponding effective potential  $V$  in this model is then given by  $\Gamma[\Phi=0, C, J, G, K, L] = \Omega V(\varphi)$ , where  $\Omega$  is the volume of  $d$ -dimensional space. That is, it can be computed from vacuum graphs with constant background field  $\varphi$ . Note that the right-hand side of Eq. (144) is the right-hand side of Eq. (93) with the one-particle reducible graphs omitted.

Except for the vacuum diagrams, we still have to use Eq. (134) to convert the graphs representing  $\Gamma$  into proper vertices. For example, to compute the one-loop contribution  $\Gamma_{123}^{(1,3)}$  to the three-point vertex, we first use Eqs. (137)–(139) to get

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \frac{1}{3} \left[ \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] = \frac{1}{4} \text{ (circle with two external lines)} + \frac{1}{6} \text{ (circle with three external lines)} \quad (145)$$

and then with Eq. (134) obtain

$$\begin{aligned} \Gamma_{123}^{(1,3)} &= -\frac{\delta^3}{\delta\Phi_1\delta\Phi_2\delta\Phi_3} \left[ \frac{1}{4} \text{ (circle with two external lines)} + \frac{1}{6} \text{ (circle with three external lines)} \right] \\ &= -\frac{1}{2} \left[ \begin{matrix} 1 \\ \diagdown \\ \text{circle} \\ \diagup \\ 2 \end{matrix} \begin{matrix} 3 \\ \text{---} \\ \text{---} \\ \text{---} \end{matrix} + \begin{matrix} 3 \\ \text{---} \\ \text{---} \\ \text{---} \\ 1 \end{matrix} \begin{matrix} 2 \\ \text{---} \\ \text{---} \\ \text{---} \end{matrix} + \begin{matrix} 2 \\ \text{---} \\ \text{---} \\ \text{---} \\ 3 \end{matrix} \begin{matrix} 1 \\ \text{---} \\ \text{---} \\ \text{---} \end{matrix} \right] - \begin{matrix} 1 \\ \text{---} \\ \text{circle} \\ \text{---} \\ 2 \end{matrix} \begin{matrix} 3 \\ \text{---} \\ \text{---} \\ \text{---} \end{matrix}. \end{aligned} \quad (146)$$

That is, each diagram with  $n$  external fields  $\Phi$  is multiplied by  $-n!$ , supplied by external arguments replacing the  $\Phi_s$ , and then splits into “crossed” graphs related by exchanging external arguments. In contrast to the case of connected Green’s functions, the external legs carry only the external arguments and do not represent free correlation functions  $G$ .

#### D. Graphs for renormalization

For the purpose of perturbatively renormalizing standard  $\phi^4$  theory, we need the 1PI Feynman diagrams representing  $\Gamma^{(L,0)}$ ,  $\Gamma^{(L,2)}$ , and  $\Gamma^{(L,4)}$  for the case  $J=K=0$ . All  $\Gamma^{(L,n)}$  with odd  $n$  are then identically zero. The recursion relation for vacuum graphs with  $L>2$  results from writing Eq. (143) for  $J=K=0$  and then making use of Eq. (78) with  $n_1=n_3=0$ ,

$$\begin{pmatrix} L \\ 0 \end{pmatrix} = \frac{1}{2(L-1)} \left[ \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} + \frac{1}{3} \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} + \frac{1}{3} \sum_{l=2}^{L-2} \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} \right]. \quad (147)$$

Notice that this is identical to Eq. (31) for vanishing two-point insertion.

For  $n=2$  and  $n=4$  we rewrite Eq. (139) with  $J=K=0$ . For  $\Gamma^{(L,2)}$  we get for  $L>1$

$$\begin{pmatrix} L \\ 2 \end{pmatrix} = \frac{1}{2} \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} + \frac{1}{6} \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} + \frac{1}{3} \sum_{l=2}^{L-1} \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array}. \quad (148)$$

while for  $\Gamma^{(L,4)}$  we get for  $L>0$

$$\begin{pmatrix} L \\ 4 \end{pmatrix} = \frac{1}{4} \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} + \frac{1}{12} \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} + \frac{1}{6} \sum_{l=2}^{L-1} \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} + \frac{1}{6} \sum_{l=1}^{L-1} \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array}. \quad (149)$$

Since now we have written down only the recursion relations without starting with the identities for  $\Gamma_l$  again, we use for the low-order terms not covered by Eqs. (147)–(149) just the results (135), (137), (138), and (142) of Sec. IV C with  $J=K=0$ ,

$$\begin{aligned} \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \bullet, & \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \frac{1}{2} \circ, & \begin{pmatrix} 2 \\ 0 \end{pmatrix} &= \frac{1}{8} \text{Diagram 1}, \\ \begin{pmatrix} 0 \\ 2 \end{pmatrix} &= \frac{1}{2} \text{Diagram 2}, & \begin{pmatrix} 1 \\ 2 \end{pmatrix} &= \frac{1}{4} \text{Diagram 3}, & \begin{pmatrix} 0 \\ 4 \end{pmatrix} &= \frac{1}{24} \text{Diagram 4}. \end{aligned} \quad (150)$$

It is now easy to use Eqs. (147)–(149) to obtain, e.g. (compare to the 1PI graphs in the tables in Ref. [3]; for the vacuum graphs, compare also with Table I in this work),

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \frac{1}{4} \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} + \frac{1}{12} \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} = \frac{1}{16} \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} + \frac{1}{48} \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array}, \quad (151)$$

$$\begin{aligned} \begin{pmatrix} 4 \\ 0 \end{pmatrix} &= \frac{1}{6} \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} + \frac{1}{18} \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} + \frac{1}{18} \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} \\ &= \frac{1}{48} \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array} + \frac{1}{24} \begin{array}{c} \text{Diagram 9} \\ \text{Diagram 10} \end{array} + \frac{1}{32} \begin{array}{c} \text{Diagram 11} \\ \text{Diagram 12} \end{array} + \frac{1}{48} \begin{array}{c} \text{Diagram 13} \\ \text{Diagram 14} \end{array}, \end{aligned} \quad (152)$$

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \frac{1}{2} \text{diagram} + \frac{1}{6} \text{diagram} = \frac{1}{8} \text{diagram} + \frac{1}{12} \text{diagram}, \quad (153)$$

$$\begin{aligned} \begin{pmatrix} 3 \\ 2 \end{pmatrix} &= \frac{1}{2} \text{diagram} + \frac{1}{6} \text{diagram} + \frac{1}{3} \text{diagram} \\ &= \frac{1}{16} \text{diagram} + \frac{1}{16} \text{diagram} + \frac{1}{24} \text{diagram} + \frac{1}{8} \text{diagram} + \frac{1}{8} \text{diagram}, \end{aligned} \quad (154)$$

$$\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \frac{1}{4} \text{diagram} + \frac{1}{12} \text{diagram} = \frac{1}{16} \text{diagram}, \quad (155)$$

$$\begin{aligned} \begin{pmatrix} 2 \\ 4 \end{pmatrix} &= \frac{1}{4} \text{diagram} + \frac{1}{12} \text{diagram} + \frac{1}{6} \text{diagram} \\ &= \frac{1}{32} \text{diagram} + \frac{1}{16} \text{diagram} + \frac{1}{8} \text{diagram}. \end{aligned} \quad (156)$$

In this way, all the graphs needed for the renormalization of  $\phi^4$  theory can be obtained (for a five-loop treatment see Refs. [8,9]). There is no need to go beyond  $L$  loop order to determine all 1PI zero-, two-, and four-point graphs through  $L$  loops. I have written a computer code implementing the recursion relations for the 1PI graphs. If we restrict ourselves to the symmetric case, it reproduces the 1PI graphs and their multiplicities (trivially related to the weights, see Ref. [3]) of Tables I–III in Ref. [3] and also all relevant entries in Tables V–VII there.

**E. Absorption of tadpoles**

Here we discuss the absorption of tadpoles, i.e.,  $\Phi$ -independent subdiagrams of the form

$$\text{diagram} \quad (157)$$

into the propagator for diagrams representing the proper vertices  $\Gamma^{(L,n)}$  with  $n > 0$  in the theory. For standard  $\phi^4$  theory, this amounts to an absorption of momentum-independent propagator corrections into the mass. This drastically reduces the amount of remaining diagrams and therefore simplifies the bookkeeping for higher-loop calculations [10].

Let us first indicate the changes to be introduced into the treatment of the asymmetric case to arrive at recursion relations for the  $\Gamma^{(L,n)}$  in the presence of a two-point insertion as defined in Sec. II. Since Eqs. (55) and (56) receive the additional terms

$$\int_2 \Delta_{12} \frac{\delta W}{\delta J_2} \quad (158)$$

and

$$2 \int_3 \Delta_{13} \frac{\delta W}{\delta G_{23}^{-1}} \quad (159)$$

on their respective right-hand sides, the changes on the right-hand sides of Eqs. (108) and (109) is the addition of the terms

$$- \int_2 \Delta_{12} \Phi_2 \quad (160)$$

and

$$- 2 \int_3 \Delta_{13} \frac{\delta \Gamma}{\delta G_{23}^{-1}} = - \int_3 \Delta_{13} \left[ (G_{23} + \Phi_2 \Phi_3) + 2 \frac{\delta \Gamma_I}{\delta G_{23}^{-1}} \right], \quad (161)$$

respectively, where we have used Eq. (118). This leads to the addition of

$$-\int_{12} \Delta_{12} \Phi_1 \Phi_2 \tag{162}$$

and

$$-\int_{12} \Delta_{12} \Phi_1 \Phi_2 - \int_{12} G_{12} \Delta_{12} + 2 \int_{12} \Delta_{12} G_{13} G_{24} \frac{\delta \Gamma_I}{\delta G_{34}} \tag{163}$$

to the right-hand sides of Eqs. (123) and (124), respectively. Then, in a notation which by now should be obvious, the right-hand sides of Eqs. (131) and (132) receive the addition of

$$\text{---} \square_{\Delta} \text{---} \tag{164}$$

and

$$\text{---} \square_{\Delta} \text{---} + \square_{\Delta} + 2 \square_{\Delta} \Gamma_I, \tag{165}$$

respectively. Note that for  $K = \Phi = 0$  and  $\Gamma_I \rightarrow -W_I$ , the second resulting equation is identical to Eq. (18).

Equations (133)–(137) remain unchanged, while the right-hand side of Eq. (138) receives the addition of

$$\frac{1}{2} \text{---} \square_1 \text{---} . \tag{166}$$

For  $n \neq 2$ , Eq. (139) remains unchanged. For  $n = 2$  with  $L > 1$ , the right-hand side of Eq. (139) receives the addition of

$$\text{---} \square_L \text{---} . \tag{167}$$

Finally, the right-hand sides of Eqs. (142) and (143) receive the additions of

$$\frac{1}{2} \square_1 \tag{168}$$

and

$$\frac{1}{2} \square_{L-1} + \sum_{l=1}^{L-2} \square_l \square_{L-l}, \tag{169}$$

respectively.

It is not hard to see then that with and only with the choices

$$\text{---} \square_1 \text{---} = -\frac{1}{2} \text{---} \circ \tag{170}$$

at the one-loop level and

$$\text{---} \square_L \text{---} = - \text{---} \circ \tag{171}$$

for  $L > 1$ , all tadpole corrections to propagators in 1PI  $n$ -point functions with  $n > 0$  will be canceled. This drastically reduces the number of diagrams to be considered at higher loop orders. Notice that with Eq. (122), the insertions (170) and (171) can be summarized by writing

$$\Delta_{12} = -\frac{1}{2} \int_{34} L_{1234} P_{34} |_{\Phi=0}, \tag{172}$$

i.e., by inserting the full propagator into a one-loop tadpole [compare to Eq. (103) and the comment following it]. It turns out that this cancellation is also true for

$$\text{---} \circ \tag{173}$$

but not for

$$\circ \tag{174}$$

i.e., not for the vacuum graphs with their proper weights. A simple diagrammatical explanation for this failure is that the combinatorics do not work out since as a matter of principle it is undefined which part of a vacuum diagram with a cutvertex (a vertex which connects two otherwise unconnected parts of a diagram) is the tadpole and which part is the rest of the diagram. A reflection of this problem was already encountered in Sec. IID, where the two-loop diagram (42) survived our one-loop resummation. Let us emphasize that the values (170) and (171) for the two-point insertions have to be used *after* evaluating the recursion relations.

Let us now establish the connection between our resummation above and the one used in Ref. [10]. In that work, a distinction is established between  $\Phi$ -independent subdiagrams of the form (157), called “snail diagrams” there, and  $\Phi$ -independent subdiagrams of the forms

$$\text{---} \circ \text{---} , \quad \text{---} \circ \text{---} , \tag{175}$$

called “tadpole diagrams” there. Reference [10] uses the usual Schwinger-Dyson equations to adjust the triple coupling and mass such that there are no more graphs of the  $n$ -point functions with  $n > 1$  to consider for the effective action (equivalent to the effective energy in our treatment) that contain any “snail” or “tadpole” subdiagrams.

One notices that graphs of the form (175) are absent altogether in our treatment of the effective energy, which contains only truly 1PI diagrams in contrast to a weaker defini-

tion of one-particle irreducibility used in Ref. [10], which allows also for  $\Phi$ -independent subdiagrams of the form (175). This absence can be traced to the fact that we work with a general background field  $\varphi$  in Eq. (48). For the computation of scattering processes,  $\varphi$  has to be adjusted to the radiatively corrected vacuum expectation value  $v$  of  $\phi$ , i.e., the true minimum of the effective potential, whose shift from the tree-level value  $v_0$  can perturbatively be computed as a sum  $v = v_0 + \text{corrections}$ . If we expanded  $v$  subsequently in our graphs, the corrections would lead to exactly the diagrams containing ‘‘tadpole diagrams’’ as subdiagrams used as a starting point in Ref. [10]. In other words, the formalism we use already takes care of the resummation of all tadpoles (175) in the effective action, so that the triple coupling and the mass experience an appropriate correction when computing the corrections to  $v_0$  and setting  $\varphi = v$ . This has nothing to do with our recursion relations, but could have been used by the authors of Ref. [10] from the start as well.

For the other class of subdiagrams, the ‘‘snail diagrams’’ (157), our result (172) agrees with the result of Ref. [10] that the sum of all such subdiagrams amounts to a full propagator in a one-loop ‘‘snail diagram’’ and that therefore an appropriate split of the mass term in standard  $\phi^4$  theory will achieve a cancellation of all such ‘‘snail diagrams.’’

## V. DISCUSSION

In this work we have derived efficient recursion relations to generate connected and 1PI Feynman diagrams for  $\phi^4$  theory both with and without  $\phi \rightarrow -\phi$  symmetry. Although we used also external sources  $J$  and field expectations  $\Phi$  as functional variables, we were able to keep the recursion relations simple by using as much as possible the free propagator  $G$  as a functional variable.

Taking  $W$  as functional of both  $G$  and  $J$  and  $\Gamma$  as functional of both  $G$  and  $\Phi$  allowed us to combine the advantages of both the ‘‘current approach’’ and the ‘‘kernel approach’’ [3]: By considering diagrams with arguments  $J$  and  $\Phi$  on the external legs we avoided having to deal with ‘‘crossed’’ diagrams which are related by exchanging external arguments on their legs. This helps keep the number of diagrams at intermediate steps low. Only when we finally want to convert the coefficient functions of  $W$  and  $\Gamma$  (in an expansion in powers of  $J$  and  $\Gamma$ , respectively) into Greens functions as in Eq. (94) or Eq. (146) do we have to consider ‘‘crossed’’ diagrams.

The applications of the recursion relations lie potentially in both statistical and particle physics. Together with a powerful numeric integration method, the relations could be used to push the computation of critical exponents in three dimensions to higher loop orders, see, e.g., Refs. [9,11,12].

Similar recursion relations can be set up for theories with other field contents as well. They are a convenient starting point for the investigation of resummations of classes of Feynman diagrams. Simple one-loop and multiloop tadpole resummation examples were given in Secs. IID and IVE, respectively. Since the identities from which the recursion relations are derived are nonperturbative, they might also be useful for other expansions than the ones organized by the number of loops or powers of coupling constants. Another field for future investigations is the systematic solution of

recursion relations for Legendre transforms of higher order than the effective energy [1,2,13]. Also, the exploitation of derivatives with respect to tensors representing interactions as in Ref. [4] seems promising to further simplify identities and recursion relations.

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## APPENDIX A: DERIVATIVES WITH RESPECT TO SYMMETRIC $G$ AND $G^{-1}$

The basic properties of derivatives with respect to an unconstrained tensor  $H_{12}$  and its inverse  $H_{12}^{-1}$  are

$$\left[ \frac{\delta}{\delta H_{12}}, \frac{\delta}{\delta H_{34}} \right] = \left[ \frac{\delta}{\delta H_{12}^{-1}}, \frac{\delta}{\delta H_{34}^{-1}} \right] = 0 \quad (\text{A1})$$

and

$$\frac{\delta H_{12}}{\delta H_{34}} = \frac{\delta H_{12}^{-1}}{\delta H_{34}^{-1}} = \delta_{13} \delta_{24}, \quad (\text{A2})$$

where, according to our conventions, the labels could mean discrete as well as continuous variables and the  $\delta$ 's are an according combination of Kronecker  $\delta$ 's and Dirac  $\delta$  functions. From

$$\begin{aligned} 0 &= \frac{\delta}{\delta H_{34}} \delta_{12} = \frac{\delta}{\delta H_{34}} \int_5 H_{15}^{-1} H_{52} \\ &= \int_5 \frac{\delta H_{15}^{-1}}{\delta H_{34}} H_{52} + \int_5 H_{15}^{-1} \frac{\delta H_{52}}{\delta H_{34}} \\ &= \int_5 \frac{\delta H_{15}^{-1}}{\delta H_{34}} H_{52} + H_{13}^{-1} \delta_{24} \end{aligned} \quad (\text{A3})$$

we get

$$\frac{\delta H_{12}^{-1}}{\delta H_{34}} = -H_{13}^{-1} H_{42}^{-1} \quad (\text{A4})$$

and therefore

$$\frac{\delta}{\delta H_{12}} = \int_{34} \frac{\delta H_{34}^{-1}}{\delta H_{12}} \frac{\delta}{\delta H_{34}^{-1}} = - \int_{34} H_{31}^{-1} H_{24}^{-1} \frac{\delta}{\delta H_{34}^{-1}}. \quad (\text{A5})$$

By exchanging  $H$  and  $H^{-1}$  in the derivation of Eqs. (A4) and (A5) we get

$$\frac{\delta H_{12}}{\delta H_{34}^{-1}} = -H_{13} H_{42} \quad (\text{A6})$$

and

$$\frac{\delta}{\delta H_{12}^{-1}} = \int_{34} \frac{\delta H_{34}}{\delta H_{12}^{-1}} \frac{\delta}{\delta H_{34}} = - \int_{34} H_{31} H_{24} \frac{\delta}{\delta H_{34}}. \quad (\text{A7})$$

When considering symmetric tensors  $G$  and  $G^{-1}$ , we have to define what we mean by derivatives with respect to them. While Eqs. (A2), (A4), and (A6) obviously need appropriate symmetrizations, we would like to keep Eqs. (A1), (A5), and (A7) untouched.

Let us for the following considerations keep  $H$  unconstrained and define  $G$  to be its symmetric part:

$$G_{12} \equiv \frac{1}{2}(H_{12} + H_{21}). \quad (\text{A8})$$

Define the derivative with respect to  $G$  by

$$\frac{\delta}{\delta G_{12}} \equiv \frac{1}{2} \left( \frac{\delta}{\delta H_{12}} + \frac{\delta}{\delta H_{21}} \right), \quad (\text{A9})$$

so that with Eq. (A1) immediately follows

$$\left[ \frac{\delta}{\delta G_{12}}, \frac{\delta}{\delta G_{34}} \right] = 0. \quad (\text{A10})$$

Then, if  $\delta/\delta G_{12}$  acts on a functional that depends on  $H$  only through  $G$ , it acts exactly to remove an appearance of  $G$  in a symmetric way,

$$\frac{\delta G_{12}}{\delta G_{34}} = \frac{1}{4} \left( \frac{\delta}{\delta H_{34}} + \frac{\delta}{\delta H_{43}} \right) (H_{12} + H_{21}) = \frac{1}{2} (\delta_{13} \delta_{24} + \delta_{14} \delta_{23}). \quad (\text{A11})$$

We also need derivatives with respect to  $G^{-1}$ , which is also symmetric in its indices. Since in general the symmetrized version of  $H^{-1}$  is not identical to  $G^{-1}$ , it turns out to be inconvenient to define derivatives with respect to  $G^{-1}$  by just replacing  $G$  and  $H$  by  $G^{-1}$  and  $H^{-1}$  in Eq. (A9), respectively. Define instead

$$\frac{\delta}{\delta G_{12}^{-1}} \equiv - \int_{34} G_{13} G_{24} \frac{\delta}{\delta G_{34}}, \quad (\text{A12})$$

which trivially implies

$$\frac{\delta}{\delta G_{12}} \equiv - \int_{34} G_{13}^{-1} G_{24}^{-1} \frac{\delta}{\delta G_{34}^{-1}}. \quad (\text{A13})$$

Using Eqs. (A10) and (A11) it is easy to check that

$$\left[ \frac{\delta}{\delta G_{12}^{-1}}, \frac{\delta}{\delta G_{34}^{-1}} \right] = 0. \quad (\text{A14})$$

Using that

$$\begin{aligned} 0 &= \frac{\delta}{\delta G_{34}} \delta_{12} = \frac{\delta}{\delta G_{34}} \int_5 G_{15}^{-1} G_{52} \\ &= \int_5 \frac{\delta G_{15}^{-1}}{\delta G_{34}} G_{52} + \int_5 G_{15}^{-1} \frac{\delta G_{52}}{\delta G_{34}} \\ &= \int_5 \frac{\delta G_{15}^{-1}}{\delta G_{34}} G_{52} + \frac{1}{2} (G_{13}^{-1} \delta_{24} + G_{14}^{-1} \delta_{23}) \end{aligned} \quad (\text{A15})$$

and therefore

$$\frac{\delta G_{12}^{-1}}{\delta G_{34}} = -\frac{1}{2} (G_{13}^{-1} G_{24}^{-1} + G_{14}^{-1} G_{23}^{-1}), \quad (\text{A16})$$

we get

$$\begin{aligned} \frac{\delta G_{12}^{-1}}{\delta G_{34}^{-1}} &= - \int_{56} G_{35} G_{46} \frac{\delta G_{12}^{-1}}{\delta G_{56}} \\ &= \frac{1}{2} \int_{56} G_{35} G_{46} (G_{15}^{-1} G_{26}^{-1} + G_{16}^{-1} G_{25}^{-1}) \\ &= \frac{1}{2} (\delta_{13} \delta_{24} + \delta_{14} \delta_{23}) \end{aligned} \quad (\text{A17})$$

and therefore, repeating the steps that lead to Eq. (A16) with the roles of  $G$  and  $G^{-1}$  exchanged,

$$\frac{\delta G_{12}}{\delta G_{34}^{-1}} = -\frac{1}{2} (G_{13} G_{24} + G_{14} G_{23}). \quad (\text{A18})$$

The upshot of these considerations is that we can work with symmetric  $G$  and  $G^{-1}$  in the first place if we use Eqs. (A10) and (A12)–(A14), as well as the symmetrized relations (A11) and (A16)–(A18).

## APPENDIX B: ELIMINATION OF $(\delta^2 W_I / \delta \bar{J}_1 \delta \bar{J}_2)_G$

In the course of changing variables from  $J$  to  $\bar{J}$  in Sec. III C, double derivatives with respect to  $\bar{J}$  appear. However, we want to replace these kinds of terms with derivatives with respect to  $G$  to keep the resulting recursion relations as simple as possible.

From the definition (54) of  $Z$  and  $W$  we have

$$-2 \left( \frac{\delta Z}{\delta G_{12}^{-1}} \right)_J = \left( \frac{\delta^2 Z}{\delta J_1 \delta J_2} \right)_G \quad (\text{B1})$$

and therefore

$$-2 \left( \frac{\delta W}{\delta G_{12}^{-1}} \right)_J = \left( \frac{\delta^2 W}{\delta J_1 \delta J_2} \right)_G + \left( \frac{\delta W}{\delta J_1} \right)_G \left( \frac{\delta W}{\delta J_2} \right)_G. \quad (\text{B2})$$

From

$$\left( \frac{\delta}{\delta J_1} \right)_G = \int_2 \left( \frac{\delta \bar{J}_2}{\delta J_1} \right)_G \left( \frac{\delta}{\delta \bar{J}_2} \right)_G = \int_2 G_{12} \left( \frac{\delta}{\delta \bar{J}_2} \right)_G \quad (\text{B3})$$



we get

$$\left(\frac{\delta W}{\delta J_1}\right)_G = \int_2 G_{12} \left(\frac{\delta W}{\delta \bar{J}_2}\right)_G \tag{B4}$$

and

$$\left(\frac{\delta^2 W}{\delta J_1 \delta J_2}\right)_G = \int_{34} G_{13} G_{24} \left(\frac{\delta^2 W}{\delta \bar{J}_3 \delta \bar{J}_4}\right)_G. \tag{B5}$$

Also,

$$\begin{aligned} \left(\frac{\delta W}{\delta G_{12}}\right)_J &= \left(\frac{\delta W}{\delta G_{12}}\right)_{\bar{J}} + \int_3 \left(\frac{\delta \bar{J}_3}{\delta G_{12}}\right)_J \left(\frac{\delta W}{\delta \bar{J}_3}\right)_G = \left(\frac{\delta W}{\delta G_{12}}\right)_J + \frac{1}{2} \int_3 (\delta_{13} J_2 + \delta_{23} J_1) \left(\frac{\delta W}{\delta \bar{J}_3}\right)_G \\ &= \left(\frac{\delta W}{\delta G_{12}}\right)_J + \frac{1}{2} \left(\frac{\delta W}{\delta \bar{J}_1}\right)_G J_2 + \frac{1}{2} \left(\frac{\delta W}{\delta \bar{J}_2}\right)_G J_1 \\ &= \left(\frac{\delta W}{\delta G_{12}}\right)_{\bar{J}} + \frac{1}{2} \left(\frac{\delta W}{\delta \bar{J}_1}\right)_G \int_3 G_{23}^{-1} \bar{J}_3 + \frac{1}{2} \left(\frac{\delta W}{\delta \bar{J}_2}\right)_G \int_3 G_{13}^{-1} \bar{J}_3. \end{aligned} \tag{B6}$$

Combining Eqs. (B2)–(B6) yields

$$\left(\frac{\delta^2 W}{\delta \bar{J}_1 \delta \bar{J}_2}\right)_G + \left(\frac{\delta W}{\delta \bar{J}_1}\right)_G \left(\frac{\delta W}{\delta \bar{J}_2}\right)_G = 2 \left(\frac{\delta W}{\delta G_{12}}\right)_{\bar{J}} + \frac{1}{2} \left(\frac{\delta W}{\delta \bar{J}_1}\right)_G \int_3 G_{23}^{-1} \bar{J}_3 + \frac{1}{2} \left(\frac{\delta W}{\delta \bar{J}_2}\right)_G \int_3 G_{13}^{-1} \bar{J}_3. \tag{B7}$$

From Eq. (68) we have

$$\left(\frac{\delta W_0}{\delta G_{12}}\right)_J = \frac{1}{2} G_{12}^{-1} - \frac{1}{2} \int_{34} G_{13}^{-1} G_{24}^{-1} \bar{J}_3 \bar{J}_4, \tag{B8}$$

$$\left(\frac{\delta W_0}{\delta \bar{J}_1}\right)_G = \int_2 G_{12}^{-1} \bar{J}_2, \tag{B9}$$

$$\left(\frac{\delta^2 W_0}{\delta \bar{J}_1 \delta \bar{J}_2}\right)_G = G_{12}^{-1} \tag{B10}$$

and combining this with Eqs. (57) and (B7) finally gives

$$\left(\frac{\delta^2 W_I}{\delta \bar{J}_1 \delta \bar{J}_2}\right)_G + \left(\frac{\delta W_I}{\delta \bar{J}_1}\right)_G \left(\frac{\delta W_I}{\delta \bar{J}_2}\right)_G = 2 \left(\frac{\delta W_I}{\delta G_{12}}\right)_{\bar{J}}. \tag{B11}$$

**APPENDIX C: GRAPHS FOR  $W^{(3,0)}$**

To demonstrate the use of the recursion relations for the Feynman diagrams constituting  $W$ , we compute here  $W^{(3,0)}$ . From Eqs. (92) and (89) we get

$$\text{Graph}(3,0) = \frac{3}{4} \text{Graph}(2,1) + \text{Graph}(2,0) + \frac{1}{2} \text{Graph}(2,1) + \frac{1}{3} \text{Graph}(2,0) + \frac{1}{2} \text{Graph}(1,1) \text{Graph}(2,0) \tag{C1}$$

and

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{2} \text{ (diagram)} + \text{ (diagram)} + \frac{1}{3} \text{ (diagram)}, \tag{C2}$$

respectively. With Eqs. (87) and (91) we get

$$\frac{1}{2} \text{ (diagram)} = \frac{1}{4} \text{ (diagram)}, \tag{C3}$$

$$\text{ (diagram)} = \frac{1}{4} \text{ (diagram)} + \frac{1}{8} \text{ (diagram)} + \frac{1}{4} \text{ (diagram)} + \frac{1}{4} \text{ (diagram)} \tag{C4}$$

and

$$\frac{1}{3} \text{ (diagram)} = \frac{1}{6} \text{ (diagram)} \tag{C5}$$

and thus

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{4} \text{ (diagram)} + \frac{1}{8} \text{ (diagram)} + \frac{1}{4} \text{ (diagram)} \\ + \frac{1}{4} \text{ (diagram)} + \frac{1}{4} \text{ (diagram)} + \frac{1}{6} \text{ (diagram)}. \tag{C6}$$

With Eqs. (87), (91), and (C6) we have

$$\frac{3}{2} \text{ (diagram)} = \frac{3}{8} \text{ (diagram)} + \frac{3}{16} \text{ (diagram)} + \frac{3}{8} \text{ (diagram)} \\ + \frac{3}{8} \text{ (diagram)} + \frac{3}{8} \text{ (diagram)} + \frac{1}{4} \text{ (diagram)}, \tag{C7}$$

$$\text{ (diagram)} = \frac{1}{4} \text{ (diagram)} + \text{ (diagram)} + \frac{3}{4} \text{ (diagram)} + \frac{1}{2} \text{ (diagram)} \\ + \frac{1}{4} \text{ (diagram)} + \frac{1}{4} \text{ (diagram)} + \frac{3}{4} \text{ (diagram)} + \frac{3}{4} \text{ (diagram)}, \tag{C8}$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ (diagram)} = \frac{1}{8} \text{ (diagram)} + \frac{1}{16} \text{ (diagram)} + \frac{1}{8} \text{ (diagram)} + \frac{1}{8} \text{ (diagram)}, \tag{C9}$$

$$2 \text{ (diagram)} = \frac{1}{4} \text{ (diagram)} + \frac{1}{2} \text{ (diagram)} + \text{ (diagram)} + \frac{1}{2} \text{ (diagram)}, \tag{C10}$$



and therefore

$$\begin{array}{c} 3 \\ \circ \\ 0 \end{array} = \frac{1}{16} \text{ (two circles)} + \frac{1}{48} \text{ (circle with horizontal line)} + \frac{1}{8} \text{ (circle with V-shape)} + \frac{1}{8} \text{ (circle with vertical line)} + \frac{1}{16} \text{ (two circles with arcs)} + \frac{1}{24} \text{ (circle with three lines)}.$$

(D3)

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